**ECON 521, Discussion Section 11**

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1. Let's have another look at Question 12 from PS3:

| | C | D |
|---|---|---|
| C | 2,2 | -1,3 |
| D | 3,-1 | 0,0 |

Consider an alternative to grim-trigger in which players play $C$ so long as $(C, C)$ was played in the previous period. If either player plays $D$, then both players play $D$ for $k$ subsequent periods, then revert to $(C, C)$. If a player plays $C$ when they are supposed to play $D$, it resets the counter on the punishment phase to zero, that is they play $k$ more periods of $(D, D)$ before returning to $(C, C)$. If $\delta = 0.35$, what is the minimum $k$ such that this is a SPE? In the solutions, in checking whether $k = 1$ worked, I checked the following condition:

$$2 + 2\delta \geq 3 + 0\delta$$

Where did this condition come from, and what alternative (but equivalent) conditions could we come up with?

*Solution:* That condition came from observing that we consider unilateral deviations assuming all others are playing according the proposed strategy profile. If we are considering just one punishment period, then, by deviating today, we will get a payoff of 3 today and 0 tomorrow. But then the next day, we return to $(C, c)$. So really, our deviation only affects two periods (today and tomorrow). Therefore, we can just consider those two periods in isolation and compare the payoffs from cooperating (2 today then 2 tomorrow) with those from defecting (3 today then 0 tomorrow). That's where the following equation comes from:

$$2 + 2\delta \geq 3 + 0\delta \Leftrightarrow \delta \geq \frac{1}{2}$$

Similarly, if there are two punishment periods, we can compare:

$$2 + 2\delta 2\delta^2 \geq 3 + 0\delta + 0\delta^2 \Leftrightarrow \delta \geq 0.37 \ (approx)$$

And if there are three punishment periods, we can compare:

$$2 + 2\delta 2\delta^2 + 2\delta^3 \geq 3 + 0\delta + 0\delta^2 + 0\delta^3 \Leftrightarrow \delta \geq 0.34 \ (approx)$$

The advantage to this format is that these are fairly simple equations. Of course, there are alternatives:

**Alternative Method 1:** You could just compare total discounted payoffs in the two cases, then, for $k = 1$

$$\frac{2}{1-\delta} \geq 3 + 0\delta + \delta^2 \frac{2}{1-\delta} \Leftrightarrow \delta \geq \frac{1}{2}$$

Similarly, if there are two punishment periods, we can compare:

$$\frac{2}{1-\delta} \geq 3 + 0\delta + 0\delta^2 + \delta^3 \frac{2}{1-\delta} \Leftrightarrow \delta \geq 0.37 \ (approx)$$

And if there are three punishment periods, we can compare:

$$\frac{2}{1-\delta} \geq 3 + 0\delta + 0\delta^2 + 0\delta^3 + \delta^4 \frac{2}{1-\delta} \Leftrightarrow \delta \geq 0.34 \ (approx)$$

As you can see, the answers are the same using this alternative method as they are with the method I used.

**Alternative Method 2:** You could compare total discounted payoffs but reason that if there is incentive to defect once, then, in fact, that defector should always play $D$ when they are supposed to play $C$. Therefore, comparing total payoffs between playing C when you're supposed to play $C$ and *always* playing $D$ when supposed to play $C$ gives us (for $k = 1$):

$$\frac{2}{1-\delta} \geq \frac{3}{1-\delta^2} \Leftrightarrow \delta \geq \frac{1}{2}$$

Similarly, if there are two punishment periods, we can compare:

$$\frac{2}{1-\delta} \geq \frac{3}{1-\delta^3} \Leftrightarrow \delta \geq 0.37 \ (approx)$$

And if there are three punishment periods, we can compare:

$$\frac{2}{1-\delta} \geq \frac{3}{1-\delta^4} \Leftrightarrow \delta \geq 0.34 \ (approx)$$

**Alternative Method 3:** You can of course take either of the two alternative methods above, divide everything by $(1-\delta)$ and then say you are comparing average discounted payoffs. Obviously it won't change the results!

2. Consider a Stackelberg oligopoly with three firms, P1, P2 and P3. The market inverse demand is given by $p = A - Q$ where $Q = q_1 + q_2 + q_3$. Assume $A > \frac{19}{5}$. P1 chooses $q_1$ first. Then P2, having observed $q_1$, chooses $q_2$. Then P3, having observed $q_1$ and $q_2$, chooses $q_3$. Each firm's costs of production are as follows:

   - For P1: $c(q_1) = 0$ (there are no production costs for P1)
   - For P2: $c(q_2) = q_2^2$
   - For P3: $c(q_3) = q_3 + F$ ($F$ is a positive constant).

   (a) Find the SPE outcome quantities.

   (b) For some $F$, P3 is losing money in equilibrium. Why is this plausible?

   *Solution:*

   We can solve this, as we often do, using backwards inductions. Starting from the end, P3 solves:

$$\max_{q_3} (A - q_1 - q_2 - q_3)q_3 - q_3 - F$$

   Then the FOC is:

$$A - q_1 - q_2 - 2q_3 - 1 = 0 \Leftrightarrow q_3^* = \frac{A - q_1 - q_2 - 1}{2}$$

   Next, P2 solves

$$\max_{q_2} (A - q_1 - q_2 - q_3)q_2 - q_2^2$$

   But P2 knows how P3 will best respond, so he plugs in that best response:

$$\max_{q_2} (A - q_1 - q_2 - \frac{A - q_1 - q_2 - 1}{2})q_2 - q_2^2$$

Simplifying a little gives:

$$\max_{q_2} \left( \frac{A}{2} - \frac{q_1}{2} - \frac{q_2}{2} + \frac{1}{2} \right) q_2 - q_2^2$$

Then the FOC is

$$\frac{A}{2} - \frac{q_1}{2} - 3q_2 + \frac{1}{2} = 0 \Leftrightarrow q_2^* = \frac{A - q_1 + 1}{6}$$

Finally, P1 solves:

$$\max_{q_1} (A - q_1 - q_2 - q_3) q_1$$

But he knows how P2 and P3 will best respond, so we can plug those in:

$$\max_{q_1} \left( A - q_1 - \frac{A - q_1 + 1}{6} - \frac{A - q_1 - q_2 - 1}{2} \right) q_1$$

Plugging in for $q_3$ put another $q_2$ in there, so we substitute that out also:

$$\max_{q_1} \left( A - q_1 - \frac{A - q_1 + 1}{6} - \frac{A - q_1 - \frac{A - q_1 + 1}{6} - 1}{2} \right) q_1$$

Rewriting:

$$\max_{q_1} \left( A - q_1 - \frac{A}{6} + \frac{q_1}{6} - \frac{1}{6} - \frac{A}{2} + \frac{q_1}{2} + \frac{A}{12} - \frac{q_1}{12} + \frac{1}{12} + \frac{1}{2} \right) q_1$$

Collecting terms:

$$\max_{q_1} \left( \frac{5}{12} (A - q_1 + 1) \right) q_1$$

Taking the FOC gives

$$\frac{5}{12}(A+1) - \frac{10}{12} q_1 = 0 \Leftrightarrow q_1^* = \frac{A+1}{2}$$

Then plugging that in to $q_2^*$ gives

$$q_2^* = \frac{A - q_1 + 1}{6} = \frac{A - \frac{A+1}{2} + 1}{6} = \frac{A+1}{12}$$

Then plugging both into $q_3^*$ gives

$$q_3^* = \frac{A - q_1 - q_2 - 1}{2} = q_3^* = \frac{A - \frac{A+1}{2} - \frac{A+1}{12} - 1}{2} = \frac{5A - 19}{24}$$

For (b), the key is to just view $F$ as a fixed cost. You can see this since $c(q_3 = 0) = F$. Therefore, even though it may be losing money, it will still choose to produce a positive quantity as defined above so as to minimize it's losses.