ECON 521, Discussion Section 12

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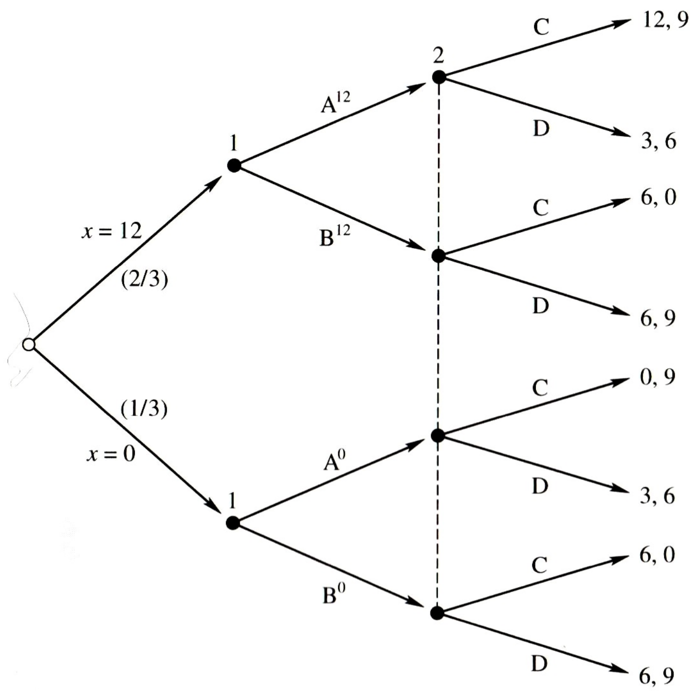
1. Consider the following Bayesian game:

1 \ 2	C	D
A	x, 9	3, 6
B	6, 0	6, 9

$$x = \begin{cases} 12 & \text{with probability } 2/3 \\ 0 & \text{with probability } 1/3 \end{cases}$$

(a) Draw the extensive form of this game, with nature making the first choice in choosing x with probabilities described above and only P1 observing nature's choice. P1 and P2 then choose between their actions simultaneously.

Solution:



(b) Draw the *ex ante* strategic form of the game.

Solution:

1 \ 2	C	D
$A^{12}A^0$	8, 9	3, 6
$A^{12}B^0$	10, 6	4, 7
$B^{12}A^0$	4, 3	5, 8
$B^{12}B^0$	6, 0	6, 9

As for where the payoffs came from, consider the top-left cell, corresponding to profile $(A^{12}A^0, C)$. In this case, there is a 1/3 chance $x = 0$ and a 2/3 chance $x = 12$ meaning payoffs are:

$$\bar{u}_1(A^{12}A^0, C) = \frac{2}{3}12 + \frac{1}{3}0 = 8$$
$$\bar{u}_2(A^{12}A^0, C) = \frac{2}{3}9 + \frac{1}{3}9 = 9$$

Looking at the cell below that, corresponding to profile $(A^{12}B^0, C)$,

$$\bar{u}_1(A^{12}B^0, C) = \frac{2}{3}12 + \frac{1}{3}6 = 10$$
$$\bar{u}_2(A^{12}B^0, C) = \frac{2}{3}9 + \frac{1}{3}0 = 6$$

And so on for all of the other cells...

(c) Solve for the Bayesian Nash Equilibrium using this form.

Solution: To solve for the BNE, we simply find the NE of the *ex ante* form. Underlining best responses gives us that the unique BNE is $(B^{12}B^0, D)$.

(d) Draw the *interim* strategic form of the game.

Solution:

C	A^0	B^0
A^{12}	12, 0, 9	12, 6, 6
B^{12}	6, 0, 3	6, 6, 0

D	A^0	B^0
A^{12}	3, 3, 6	3, 6, 7
B^{12}	6, 3, 8	6, 6, 9

Note that the playoffs for the two types of P1 here are deterministic, whereas the payoff for P2 (P3 here since row and column are occupied by the two types of P1) are expected payoffs.

(e) Solve for the Bayesian Nash Equilibrium, and confirm your answer matches that from above.

Solution: Underlining gives us (B^{12}, B^0, D) which is equivalent to the BNE we found above.

2. Consider a Cournot duopoly game with incomplete information. Demand is given by $p = 1-Q$ where Q is total quantity. Firm 1 selects q_1, which it produces at no cost. Firm 2's cost of

production (selected by nature) is private information. With probability 1/2, firm 2 produces at zero cost. With probability 1/2, firm 2 produces with marginal cost of 1/4. Call the former type of firm 2 "L" and the latter type "H" (for low and high cost, respectively). Firm 2 knows its type, whereas firm 1 knows only the probability of L and H (1/2 each). Let q_2^H and q_2^L denote the quantities selected by the two types of firm 2.

(a) If the goal is to find the BNE in this game, why is it impossible to write the *ex ante* and *interim* strategic forms as we did in the last problem?

Solution: The problem is that each firm (or type of firm, for firm 2) has an infinite action space, so we can't represent the game in the strategic form.

(b) Your answer to (a) notwithstanding, you can use the *interim* logic to find the BNE here. To do so, treat each type as a separate player. Find the best response for each and solve for the BNE.

Solution: Treating each type of firm as a distinct player, we can think of this like a three-player game and solve for each firm. Start with the low type of firm 2, which solves

$$\max_{q_2^L}(1 - q_1 - q_2^L)q_2^L$$

This gives us the first order condition and low type firm 2's best response:

$$1 - q_1 - 2q_2^L = 0 \quad \Leftrightarrow \quad q_2^{L*} = \frac{1 - q_1}{2} = \frac{1}{2} - \frac{q_1}{2} \tag{1}$$

The high type of firm 2 solves

$$\max_{q_2^H}(1 - q_1 - q_2^H)q_2^H - \frac{1}{4}q_2^H$$

This gives us the first order condition and high type firm 2's best response:

$$1 - q_1 - 2q_2^H - \frac{1}{4} = 0 \quad \Leftrightarrow \quad q_2^{H*} = \frac{1 - q_1 - \frac{1}{4}}{2} = \frac{3}{8} - \frac{q_1}{2} \tag{2}$$

Finally, since firm 1 believes it is facing a high or a low type with equal probability, it maximizes its expected profits, solving

$$\max_{q_1} \frac{1}{2}(1 - q_1 - q_2^L)q_1 + \frac{1}{2}(1 - q_1 - q_2^H)q_1$$

This gives us the first order condition and firm 1's best response:

$$\frac{1 - 2q_1 - q_2^L + 1 - 2q_1 - q_2^H}{2} = 0 \quad \Leftrightarrow \quad q_1^* = \frac{2 - q_2^L - q_2^H}{4} = \frac{1}{2} - \frac{q_2^L + q_2^H}{4} \tag{3}$$

Finally, we can solve (1), (2) and (3) as a system of three equations with three unknowns. Start by plugging (1) and (2) into (3) to solve for q_1:

$$q_1 = \frac{1}{2} - \frac{\frac{1}{2} - \frac{q_1}{2} + \frac{3}{8} - \frac{q_1}{2}}{4} \quad \Leftrightarrow \quad q_1^* = \frac{3}{8}$$

Then, plugging that back into (1) and (2) yields

$$q_2^L = \frac{5}{16} \quad \text{and} \quad q_2^H = \frac{3}{16}$$

Therefore, the BNE is where firm 1 produces $q_1 = \frac{3}{8}$ and firm 2 produces $q_2 = \frac{5}{16}$ if it learns it has low costs and $q_2 = \frac{3}{16}$ if it learns it has high costs.

3. In the lecture, we have seen that in a second-price auction with two bidders whose values are drawn randomly from a uniform distribution on $[0, 1]$, there exists a symmetric BNE in which agent i bids v_i. We have also seen that in a first-price auction in the same environment, there exists a symmetric BNE in which agent i bids $v_i/2$. Further, we found that expected revenue in either case was $1/3$. Now consider an all-pay auction in the same environment – that is, the highest bidder is awarded the good but both agents pay their bids.

   (a) What is the expected revenue from this auction? (Hint: No math required) Why? Given this, should a seller consider using an all-pay auction to sell the good?

   Solution: By the Revenue Equivalence Theorem, the expected revenue here will also be 1/3 here also. A seller could certainly consider this, given it gives him the same expected revenue as the more traditional first and second price auctions. The reason we don't see this auction much is probably more because most sellers care somewhat about the buyers and think it unfair to charge the losers – there might also be an issue of actually getting the loser to pay, though you could collect the bids beforehand.

   (b) Show that there exists a symmetric BNE in which each agent bids $b_i = \frac{1}{2}v_i^2$ and calculate the expected revenue to confirm that the Revenue Equivalence Theorem holds here.

   Solution: To verify this, we suppose that one agent chooses $b_i = \frac{1}{2}v_i^2$ and verify that, given this, the other wants to also. Supposing agent 2 is bidding $b_2 = \frac{1}{2}v_2^2$, agent 1 solves:

$$\max_{b_1} P(\text{Agent 1 Wins})v_1 - b_1$$

   But the probability that agent 1 wins is the probability that his bid is greater than agent two's bid, and we know what the equals (as a function of v_2):

$$\max_{b_1} P\left(b_1 > \frac{1}{2}v_2^2\right) v_1 - b_1$$

   Rearranging:

$$\max_{b_1} P\left(v_2 < \sqrt{2b_1}\right) v_1 - b_1$$

   And since v_2 is uniformly distributed on $[0, 1]$,

$$P\left(v_2 < \sqrt{2b_1}\right) = \int_0^{\sqrt{2b_1}} 1 \text{ d}x = \sqrt{2b_1}$$

   Which means agent 1 just solves

$$\max_{b_1} \sqrt{2b_1}v_1 - b_1$$

   Taking a FOC gives us

$$\begin{aligned} \frac{\sqrt{2}v_1}{2\sqrt{b_1^*}} - 1 &= 0 \\ 2\sqrt{b_1^*} &= \sqrt{2}v_1 \\ 4b_1^* &= 2v_1^2 \\ b_1^* &= \frac{1}{2}v_1^2 \end{aligned}$$

Since given that one agent's strategy is $b_i^* = \frac{1}{2}v_i^2$, the other wishes to play the same strategy, we have confirmed that this is a BNE. Note that in this BNE, the agents shade their bids even more than they do in the first price auction, as $\frac{1}{2}v_i^2 \leq \frac{1}{2}v_i$ since $v_i \in [0, 1]$.

Finally, we also want to check revenue equivalence here. Remember that the expected value of a continuous random variable x is just $\int_{-\infty}^{\infty} x f(x)\ \mathrm{d}x$. Therefore, the expected bid is then

$$\mathbf{E}[b_i] = \int_0^1 \frac{1}{2}v_i^2\ \mathrm{d}v_i = \left[\frac{1}{6}v_i^3\right]_0^1 = \frac{1}{6}$$

Since the seller gets the bid from each bidder, the seller's expected revenue is 1/3, just as the Revenue Equivalence Theorem would have us believe.