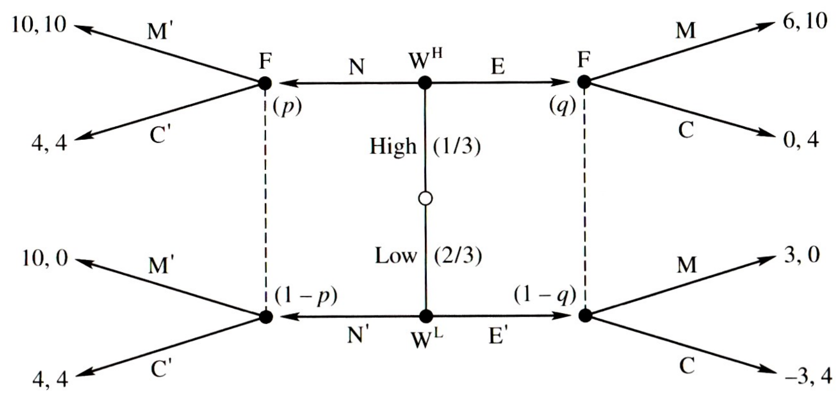
ECON 521, Discussion Section 13

TA: Shane Auerbach (sauerbach@wisc.edu) ; Date: 12/12/14

1. Consider the following job-market signaling game similar to Spence 1973 (and as presented in Watson page 350):



The incomplete information game above is played between a worker (W) and a firm (F). The worker has private information about her level of ability. With probability 1/3 she is a high type (H) and with probability 2/3 she is a low type (L) – these probabilities are common knowledge. After observing her own type, the worker decides whether to obtain a costly education (E) or not (N); think of E as getting a degree. The firm observes the worker's education (which is described in her resume), but the firm does not observe the worker's quality type. The firm then decides whether to emply the worked in an important managerial positions (M) in a much less important clerical job (C). In equilibrium, a firm may deduce the worker's quality level on the basis of the worker's education (or not). You can see from the payoffs that high and low type value the two jobs equally (10 and 4 for M and C respectively), however the two types have different educations costs (4 for H and 7 for C). The firm gets a payoff of 10 from putting an H worker into an M job, a payoff of 4 from putting any worker into a C job, and a payoff of zero from putting an L worker into a M job. Note that the firm's payoffs are independent of the education level of the workers. To find Perfect Bayesian Equilibria (PBE), we can look for pooling equilibria (in which both types of worker make the same decision) and separating equilibria (in which the two types of workers make different decisions).

(a) Find the pooling equilibrium.

Solution: If there is a pooling equilibrium, then for it be to pooling, it must be that both types are choosing education, i.e. EE', or both types choosing to forego education, i.e. NN'. Let's consider each case:

- $\underline{\sigma_{W^H} = E \text{ and } \sigma_{W^L} = E'}$. In this case, then $q = 1/3$ in a PBE. Given this $\bar{u}_F(M,q) = \frac{1}{3}10 < \bar{u}_F(C,q) = 4$, so the firm must choose C. However, then L's payoff is -3, whereas L could get at least 4 by playing N'. Hence, there exists no pooling equilibrium with $\sigma_{W^H} = E$ and $\sigma_{W^L} = E'$.
- $\underline{\sigma_{W^H} = N \text{ and } \sigma_{W^L} = N'}$. In this case, then $p = 1/3$ in a PBE. Given this $\bar{u}_2(M',p) = \frac{1}{3}10 < \bar{u}_2(C',p) = 4$, so the firm must choose C'. For this to be an equilibrium, there must be no profitable deviation. This of course depends on whether the firm is choosing M or C when observing an educated candidate. If the firm is choosing M, then H would have a profitable deviation to E. Therefore, in the pooling equilibrium, the firm must choose C. For this

to be the case, it must be that $\bar{u}_F(M,q) \leq \bar{u}_F(C,q) \Leftrightarrow 10q \leq 4 \Leftrightarrow q \leq 2/5$. Therefore, there exists a pooling equilibrium with strategies NN', CC' and $p = 1/3$ and $q \leq 2/5$.

- *An aside:* There's actually also a mixed-strategy pooling equilibrium identical to above except with $q = 2/5$, so that the firm is indifferent after observing an educated candidate and the firm still putting sufficient probability on C in a mixed strategy such that there is no incentive to deviate. To be precise, $q = 2/5$ and

$$4 > 6\sigma_F(M) + 0(1 - \sigma_F(M)) \;\;\Leftrightarrow\;\; \sigma_F(M) \leq 2/3$$

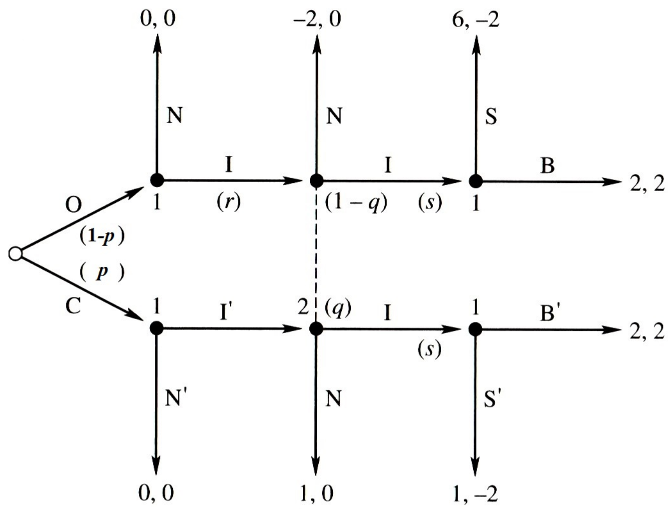
(b) Find the separating equilibrium in pure strategies.

Solution: If there is a separating equilibria, then for it to be separating, it must be that one of the types is choosing to get an education and the other is not. Therefore, there are two possibilities EN' and NE', and we check each separately.

- $\underline{\sigma_{W^H} = N \text{ and } \sigma_{W^L} = E'}$. In this case, $p = 1$ and $q = 0$ in any PBE. Given this, the Firm's optimal strategy is CM'. That is, all the educated types (who are all L types) are placed in C positions and the uneducated folk, all the H types, are put in M positions. However, in this proposed PBE, L types would have payoffs of -3 and would have an incentive to deviate (i.e. not get an education) which would yield them a payoff of 10. Therefore, there exists no PBE in this case.

- $\underline{\sigma_{W^H} = E \text{ and } \sigma_{W^L} = N'}$. In this case, $p = 0$ and $q = 1$ in any PBE. Given this, the Firms optimal strategy is MC'. That is, all educated types (who are all H types) are placed in M positions and the uneducated folk, all the L types, are put in C positions. Is there a profitable deviation? No. If Hs deviate to N, then they get C positions, giving them a payoff of 4, which is less than the 6 they get in the proposed equilibrium. If Cs deviate to E, then they get an M position and a payoff of 3, but that's less than the 4 they get in the proposed eqm. Therefore, there exists a separating PBE with strategies EN', MC' and $p = 0$ and $q = 1$.

2. Consider the following investment game with incomplete information:



The story of this game as follows. P1 owns a remote-controlled drone. It's pretty sweet, but it's broken in two ways. First, the rotor is broken – fortunately P1 knows how to fix the rotor. Second, the remote-control is not working – P1 does not know how to fix it, but P2 does. If P1 invests time in fixing the rotor (I and I'), and P2 invests time in fixing the controller (I), then the drone becomes operable. The repairs must happen in that order – you don't want to test the controller (and thus the drone) when it still has a broken rotor (duh!). If the drone is operable, P1 may choose to be *benevolent* (B) and share it with P2, or she may choose to be *selfish* (S) and keep it to herself. The added complication is that P2 does not know whether P1 is *ordinary* (O) in that she would prefer not to share or *cooperative* (C) in that she would prefer to share (and actually enjoys working on the drone). In particular, there is a probability p that P1 is cooperative, and a probability $1 - p$ she is ordinary. The payoffs are listed above.

(a) Suppose it is known that P1 is the ordinary type, i.e. $p = 0$ above. Find the subpage perfect equilibrium (SPE) of the resulting complete information game.

Solution: If $p = 0$ we can restrict attention to the top half of the game and use backwards induction to show that in the SPE, neither player invests and P1 would not share (if the investements were made).

(b) Suppose it is known that P1 is the cooperative type, i.e. $p = 1$ above. Find the SPE of the resulting complete information game.

Solution: If $p = 1$ we can restrict attention to the bottom half of the game and use backwards induction to show that in the SPE, both players invest and P1 is benevolent (i.e. shares the drone).

(c) Now suppose $p = 1/4$. Argue that there exists no perfect Bayesian equilibrium (PBE) in pure strategies.

Solution: First note that for the cooperative type has a strict incentive both to invest (I') and to be benevolent (B'). Similarly, the ordinary type has a strict incentive to be selfish S. Therefore, any PBE must specify S, I', and B'.

Given this, for P1, the only question is what the ordinary type will do at the beginning – will she invest in fixing the rotor, or choose not to? Obviously this depends on what P2 intends to do at her information set, which in turn depends upon q. In particular, given S and B', P2 prefers I to N if and only if

$$2q - 2(1-q) > 0 \quad \Leftrightarrow \quad q > 1/2$$

Now, we consider the two possibilities of the ordinary P1's first move for a PBE in pure strategies:

- Ordinary P1 plays N. Then $q = 1$. Obviously then P2 chooses I. However, given that P2 is choosing I, the ordinary P1 would have an incentive to deviate to play I as $2 > 0$. Therefore, no PBE exists where the ordinary P1 plays pure-strategy N.
- Ordinary P1 plays I. Then $q = 1/4$. Since $q = 1/4 < 1/2$, P2 prefers N to I. But given that, ordinary P1 has an incentive to deviate from I to N. Therefore, no PBE exists where the ordinary P1 plays pure-strategy I.

Since this exhausts the possibilities, we can conclude that there exists no PBE in pure strategies.

(d) Still supposing $p = 1/4$. Find a PBE in mixed strategies.

Solution: We can still use the framework we set out in the previous parts. We already showed that there exists no PBE in which ordinary P1 plays pure-strategy I or in which ordinary P1 plays pure-strategy N. But what if ordinary P1 plays a mix with probability r on I and probability $1-r$ on N. For this to be the case, it must be that ordinary P1 is indifferent between N and I, which means that the expected payoff from I must be exactly zero. This occurs only when P2 is mixing over I and N and putting probability 1/4 on I. Therefore s (as shown in the extensive form) must be 1/4.

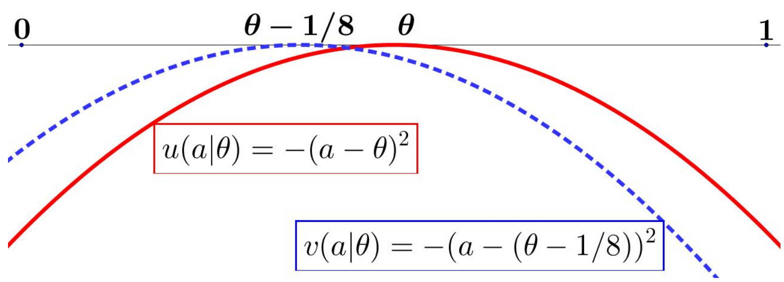
Further, P2 will only be willing to randomize if she is indifferent between her two options, i.e. her expected utility from I is exactly zero. For this to be the case, P2's belief must put equal weight on the two types of P1, i.e. $q = 1/2$. But we're looking for a Perfect **Bayesian** equilibrium, so q must be consistent with Bayes' rule:

$$q = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{3}{4}r}$$

Since we've already said $q = 1/2$, this implies $r = 1/3$. And this gives us the PBE in mixed strategies (our unique PBE) in which P1 uses the strategy specifying S, I' and B' with certainty and action I with probabily 1/3. P2 holds beliefs $q = 1/2$ and chooses I with probability 1/4.

3. In the previous Spence signaling game, the signal (education there) could be informative precisely because it was more costly for one type of agent than for the other. But what about when the signal is costless? We call these models of *cheap talk*, pioneered by the classic paper of Crawford and Sobel (1982). The following question is a simplification of that model:

   Define $\theta \in [0,1]$ to represent the degree to which the Hudson River is polluted. In Albany, New York, the governor asks a bureaucrat to work out how polluted the Hudson is – that is, she is asked to determine θ. The bureaucrat will then report a to the governor, and she wants to be as right as possible, so her utility is given by $u(a|\theta) = -(a-\theta)^2$. The problem is that the bureaucrat does not know θ or have any means to measure it directly. In particular, her prior belief is that all levels of $\theta \in [0,1]$ are equally likely. Fortunately, she has an expert acquaintance who knows how to measure θ. The expert, however, also consults with some big factories that pollute the river, and therefore has a slight bias in that if the true level of pollution is θ, he would prefer the governor thought it was $\theta - 1/8$. The expert's utility can therefore be represented by $v(a|\theta) = -(a-(\theta-1/8))^2$. He shares the bureaucrat's prior belief. All of the above is common knowledge to both the bureaucrat and the expert, and is represented below (the $\theta = 1/2$ case is shown):



   The expert may measure θ and then communicate (with message m) his findings in whatever way he wants to the bureaucrat.

   (a) Argue (simply) that there cannot be a PBE in which the expert reports the true θ.

   Solution: If the expert reports $m = \theta$, then obviously the bureaucrat will report $a = m$ to the governor. But then the expert has an incentive to deviate to $m = \theta - \frac{1}{8}$.

   (b) Argue (simply) that there cannot be a PBE in which the expert reports $\theta - 1/8$.

   Solution: If the expert reports $m = \theta - 1/8$, then in equilibrium the bureaucrat will report $a = m + 1/8$ to the governor. But then the expert has an incentive to deviate to $m = \theta - \frac{2}{8}$.

   (c) There is a PBE in which the expert truthfully reports that θ is either above ($m = H$) or below ($m = L$) a certain threshold θ^* and the bureaucrat believes the report. Find θ^*.

   Solution: Notice that the bureaucrat maximizes her utility by telling the governor $a = \mathbf{E}[\theta]$. Let L denote the expert report that $\theta < \theta^*$ and let H denote the expert report that $\theta > \theta^*$. Since we assume the bureaucrat believes the expert in this PBE (and he's reporting truthfully), it follows that $\mathbf{E}[\theta|H] = \frac{\theta^*+1}{2}$ and $\mathbf{E}[\theta|H = L] = \frac{\theta^*}{2}$.

   If the expert is truthfully reporting whether H and L. Then it must be that he prefers to report L when $\theta < \theta^*$ and H when $\theta > \theta^*$. In particular, he must be indifferent when $\theta = \theta^*$. We can use this fact, along with his payoffs, to work out θ^*.

$$\begin{aligned}
v(a|\theta^*, H) &= v(a|\theta^*, L) \\
-(a_H - (\theta^* - 1/8))^2 &= -(a_L - (\theta^* - 1/8))^2 \\
-\left(\frac{\theta^* + 1}{2} - \left(\theta^* - \frac{1}{8}\right)\right)^2 &= -\left(\frac{\theta^*}{2} - \left(\theta^* - \frac{1}{8}\right)\right)^2 \\
\left(\frac{\theta^* + 1}{2} - \left(\theta^* - \frac{1}{8}\right)\right)^2 &= \left(\frac{\theta^*}{2} - \left(\theta^* - \frac{1}{8}\right)\right)^2 \\
\theta^* &= \frac{3}{4}.
\end{aligned}$$

To complete the argument, we would also want to show that the expert wants to report L given $\theta < 3/4$ and H given $\theta > 3/4$. You can check this for yourselves.

Aside: A few other interesting notes about this model:

- There is always a PBE in which the expert's signal is ignored. We call these *babbling* equilibria.
- Here, we had a *bias* of 1/8. It turns out that for any bias less than 1/4 in this environment, there exists at least one equilibrium which partitions the state space in the same way we have done here. For smaller and smaller biases, the $[0, 1]$ state space can be partitioned into more and more chunks in equilibrium.
- For any bias, no matter how small, there is information loss in that the state space can only be partitioned into finitely many chunks. That is, for any positive bias, there is no eqm. in which the expert reports a specific number.
- The influential equilibria (where the expert's signal effectively partitions the state space) are better *ex ante* for both agents in that they have higher expected utility than in the babbling equilibrium. It is not necessarily true that they are both better off *ex post*. For an example of the latter, consider what happens here if $\theta = 1/2$ – then the bureaucrat is better off in the babbling equilibrium then she is in the influential equilibrium.
- See Crawford and Sobel (1982) and any of the billion subsequent papers on cheap talk for more information!