ECON 521, Discussion Section 3

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Q1 (Properties of the Rationalization Operator, ρ)

Consider the extended Battle of The Sexes Game below as an example.

Extended BoS :

	b	s	c
B	4,3	0,2	0,0
S	0,1	3,4	0,0
C	1,1	1,2	5,0

State whether each of the following statements is true or false and explain using the game above.

$a : \forall C \subseteq A, \rho(C) \subseteq (C)$

$b : \forall C \subseteq A, |\rho(C)| \leq |C|$ (you may need to use a different game to show this one)

$c : \forall i \in \mathbb{N}, \rho^i(A) \subseteq \rho^{i-1}(A)$

$d : a_i$ is rationalizable if and only if $\forall C \subseteq A$ such that $a_i \in C, \rho(C) = C$

Solution:

- a is false. For example, if $C = \{B\} \times \{s\}$ then $\rho(C) = \{S\} \times \{b\} \not\subseteq C$. So if we apply the rationalizability operator to an arbitrary subset of the action space, the output might not be a subset of that arbitrary subset.
- b is false also. I don't think you can actually show this in the game above (sorry about that), but consider this awful awful awful game:

  Italian "Football" :

	l	r
T	0,0	0,0
B	0,0	0,0

  Then if $C = \{T\} \times \{l\}$ then $\rho(C) = \{T, B\} \times \{l, r\}$ and $|\rho(C)| \not\leq |C|$. So not only is ρ necessarily outputting a subset of the input, it might even be outputting a larger set than the input!
- c is true. This is the good news. If you start with the full game and keep applying the rationalization operator, the output will always be a weak subset of the input. This comes from the monotonicity of ρ. Note also that this implies that the cardinality of the output will be weakly less than the cardinality of the input (the point mentioned in b). So, basically, ρ's monotonicity means it has this nice nested set property when you're applying it iteratively to A. But if you start it on an arbitrary subset of the full action space, it doesn't necessarily have that nested set property.
- d is false. It is a little unrelated to the points above, but still important. The key is that for an action to be rationalizable, it has to belong to at least one set $C \subseteq A$ such that $\rho(C) = C$. It doesn't have to be that for all sets C that include the action, $\rho(C) = C$. It's another issues with quantifiers. As an example above, you'll note that if $C = \{B\} \times \{s\}$, then $\rho(C) = \{S\} \times \{b\} \neq C$. However, this doesn't mean that B and s are not rationalizable. In fact, they also belong to the set $C' = \{B, S\} \times \{b, s\}$ where $\rho(C') = C'$, so they are rationalizable.

Q2 (Osborne & Rubinstein Exercise 64.1): Each of two players announces a nonnegative integer equal to at most 100. If $a_1 + a_2 \leq 100$, then each player i receivers a payoff of a_i. If $a_1 + a_2 > 100$, and $a_i < a_j$, then player i receives payoff of a_i and player j receives $100 - a_i$; if

$a_1 + a_2 > 100$ and $a_i = a_j$ then each player receives 50. Show that the game is dominance solvable and find the set of surviving outcomes. Here, you may eliminate weakly dominated strategies in solving the game, though this is dicey in general.

Solution: $A_i = [0, 100] \cap \mathbb{Z}$. Any $a_i < 50$ is strictly dominated by $a_i = 50$:

a_{-i}	$u_i(a_i = 50)$	$u_i(a_i < 50)$
$a_{-i} > 50$	50	$a_i < 50$
$a_{-i} \leq 50$	50	$a_i < 50$

And 50 is weakly dominated by 51 and 100 is weakly dominated by 99 (check for yourself).

$$A = [0, 100] \cap \mathbb{Z} \times [0, 100] \cap \mathbb{Z}$$
$$ND^1(A) = [51, 99] \cap \mathbb{Z} \times [51, 99] \cap \mathbb{Z}$$

Now, 51 is the unique best response to 52, so 51 is not weakly dominated. However, given that 100 has been eliminated from consideration, 98 weakly dominates 99. Similarly, once 99 is eliminated from consideration, 97 weakly dominates 98. And this process continues until only 51 remains.

$$ND^2(A) = [51, 98] \cap \mathbb{Z} \times [51, 98] \cap \mathbb{Z}$$
$$ND^3(A) = [51, 97] \cap \mathbb{Z} \times [51, 97] \cap \mathbb{Z}$$
$$\ldots$$
$$ND^{49}(A) = \{51\} \times \{51\}$$

Since we got down to a single profile of actions, we can say that this game is dominance solvable (but we did need to use weak dominance).

Q3 (Osborne Exercise 372.9): Give an example of an action that is a best response to a belief but is not a best response to any belief that assigns probability 1 to a single action. One way to do this is to design your own static game such that this is true.

Solution: Consider the following game. If $\mu^1(l) = \mu^1(r) = \frac{1}{2}$, then M is the best response, even though M is not a best response to either l or r.

Only P1 Payoffs Shown :

	l	r
T	3	0
M	2	2
B	0	3

Q4 (Watson Exercise 6.1): Find the set of rationalizable actions for each player in **G**.

G :

	W	X	Y	Z
U	3,6	4,10	5,0	0,8
M	2,6	3,3	4,10	1,1
D	1,5	2,9	3,0	4,6

Solution: To find the set of rationalizable strategies, we see what survives iterated deletion of strictly dominated strategies. First, note that X strictly dominates Z, so we can eliminate Z from consideration.

G :

	W	X	Y
U	3,6	4,10	5,0
M	2,6	3,3	4,10
D	1,5	2,9	3,0

Now note that M strictly dominates D, so we remove D from consideration. But actually U also strictly dominated both M and D, so we can eliminate both. And given that, the best response to U for P2 is X. Therefore, only U and X are rationalizable.

Q5 (Hotelling's Model of Electoral Competition): Suppose there are two candidates in an election and the set of possible positions they can take is $\{0, 1, 2, \ldots, l\}$, where l is even. Each voter has his own position, and will vote for whichever candidate is closest to his position. Suppose there exists a position m such that exactly half of the voters' positions are at least m and exactly half of the voters' positions are at most m. Each candidate wants to win the election by getting the most votes – they only care about winning, not how many votes they get. If the two candidates get the same number of votes, they both lose. What actions survive iterated deletion of weakly dominated strategies?

Solution: This is actually not the best question, because you can argue immediately that choosing m as your position weakly dominates any other. So you don't need iteration.

	$u_i(a_i < m)$	$u_i(a_i = m)$	$u_i(a_i > m)$
$a_{-i} < m$	Win or Lose	Win	Win or Lose
$a_{-i} = m$	Lose	Lose	Lose
$a_{-i} > m$	Win or Lose	Win	Win or Lose

From the chart above, you can see $a_i = m$ weakly dominates other strategies.

Q6 (Watson Exercise 8.7): Consider a game in which, simultaneously, player 1 selects a number $x \in [2, 8]$ and player 2 selects a number $y \in [2, 8]$. The payoffs are given by

$$u_1(x, y) = 2xy - x^2$$
$$u_2(x, y) = 4xy - y^2$$

Calculate the justifiable and rationalizable actions for each player in this game.

Solution: P1 solves

$$\max_x 2xy - x^2.$$

Taking the FOC gives $x^* = y$ which is a best response function. P2 solves

$$\max_y 4xy - y^2.$$

Taking the FOC gives $y^* = 2x$. Given that P1 wants to match P2 and P2 wants to double P1, we get the following

$$A = [2, 8] \times [2, 8]$$
$$\rho^1(A) = [2, 8] \times [4, 8]$$
$$\rho^2(A) = [4, 8] \times [4, 8]$$
$$\rho^3(A) = [4, 8] \times \{8\}$$
$$\rho^4(A) = \{8\} \times \{8\}$$

The justifiable actions are those included in $\rho^1(A)$ and the rationalizable actions are just 8 for each player.