ECON 521, Discussion Section 4

TA: Shane Auerbach (sauerbach@wisc.edu) ; Date: 9/26/14

1. Find all mixed strategy equilibria (including pure-strategy NE) of the following games:

	L	M	R
U	8, 3	3, 5	6, 3
C	3, 3	5, 5	4, 8
D	5, 2	3, 7	4, 9

G

	L	M	R
U	8, 1	0, 2	4, 3
C	3, 1	4, 4	0, 0
D	5, 0	3, 3	1, 4

G′

Solution:

Let's start with **G**. First, note that since there are three actions for player 1, there are technically seven possible supports for his strategy: UCD, UC, UD, CD, U, C, D. Similarly, there are are seven possible supports for player 2. Therefore, there are in fact 49 different combinations of mixed strategies that we could consider in looking for mixed equilibria. Since that sounds like an unpleasant exercise, let's try to narrow it down.

First note that M strictly dominates L for player 2. Also, a 50/50 mix of U and C strictly dominates D for player 1. Then, we are left with:

	M	R
U	$3, \underline{5}$	$\underline{6}, 3$
C	$\underline{5}, 5$	$4, \underline{8}$

First, the underlining above for best responses shows that there is no pure-strategy NE. Looking at mixed equilibria, since, for each player, the best response to each of his opponents actions is unique, neither player wants to mix unless the other is mixing. Therefore, both plays must mix to make the other indifferent. Let P1 put probability p on U, i.e. $\alpha_1(U) = p$ and probability $1-p$ on C. Let P2 put probability q on M, i.e. $\alpha_2(M) = q$ and probability $1-q$ on R.

$$\text{To make P2 indiff:} \qquad 5p + 5(1-p) = 3p + 8(1-p) \qquad \Leftrightarrow p = \frac{3}{5}$$

$$\text{To make P1 indiff:} \qquad 3q + 6(1-q) = 5q + 4(1-q) \qquad \Leftrightarrow q = \frac{1}{2}$$

Therefore, the unique mixed equilibrium is $\left((\frac{3}{5}, \frac{2}{5}, 0), (0, \frac{1}{2}, \frac{1}{2})\right)$

Now to **G′**. Again, M strictly dominates L. But we can go no further with iterated deletion. To see this, note that U and C cannot be strictly dominated since they are best responses to R and M respectively. While D is not a best response to either pure action, its average is equal to the average of U and C, so a mix of U and C won't strictly dominate it. Uh oh! So we're working with:

	M	R
U	$0, 2$	$\underline{4}, \underline{3}$
C	$\underline{4}, \underline{4}$	$0, 0$
D	$3, 3$	$1, \underline{4}$

Let's underline the best responses to find the pure-strategy NE. That gives us NE of CM and UR. Maybe that's all of the mixed equilibria? Probably not – it turns out that there are quite often an odd number of equilibria, so whenever you find two pure-strategy NE, you should expect to find at least one additional mixed equilibrium also.

Since, for both players, each opponent action has a unique best response, there cannot exist any equilibria in which one player mixes and the other doesn't. The reason is the player mixing wouldn't want to mix because he has a unique best response to the other players actions. Anyway, this still allows for three possibilities.

Possibility 1: 1 mixes on U and C, 2 mixes on M and R Let p be the probability of U and q be the probability on M

$$\text{To make P2 indiff:} \qquad 2p + 4(1-p) = 3p + 0(1-p) \qquad \Leftrightarrow p = \frac{4}{5}$$

$$\text{To make P1 indiff:} \qquad 0q + 4(1-q) = 4q + 0(1-q) \qquad \Leftrightarrow q = \frac{1}{2}$$

So we would propose as an equilibrium $\left((\frac{4}{5}, \frac{1}{5}, 0), (0, \frac{1}{2}, \frac{1}{2})\right)$

Don't forget to check: P1 might still prefer to play D given P2's mix. P1's expected utility in this mix is $\frac{4}{5}\frac{1}{2}4 + \frac{1}{5}\frac{1}{2}4 = 2$. If he played D against this mix, then his expected utility would be $\frac{1}{2}3 + \frac{1}{2}1 = 2$. Since his utility from the proposed mix is at least as high as his utility from the other option (D), the proposed mix is a mixed equilibrium.

Possibility 2: 1 mixes on U and D, 2 mixes on M and R Let p be the probability of U and q be the probability on M

$$\text{To make P2 indiff:} \qquad 2p + 3(1-p) = 3p + 4(1-p) \qquad \Leftrightarrow ※$$

Contradiction! That equation can't hold. Why not? Because R is better than M given U and given D, so if P1 is mixing over U and D, P2 will only want to play R.

Possibility 3: 1 mixes on C and D, 2 mixes on M and R Let p be the probability of C and q be the probability on M

$$\text{To make P2 indiff:} \qquad 4p + 3(1-p) = 0p + 4(1-p) \qquad \Leftrightarrow p = \frac{1}{5}$$

$$\text{To make P1 indiff:} \qquad 4q + 0(1-q) = 3q + 1(1-q) \qquad \Leftrightarrow q = \frac{1}{2}$$

So we would propose as an equilibrium $\left((0, \frac{1}{5}, \frac{4}{5}), (0, \frac{1}{2}, \frac{1}{2})\right)$

Don't forget to check: P1 might still prefer to play U given P2's mix. P1's expected utility in this mix is $\frac{1}{5}\frac{1}{2}4 + \frac{4}{5}\frac{1}{2}3 + \frac{4}{5}\frac{1}{2}1 = 2$. If he played U against this mix, then his expected utility would be $\frac{1}{2}0 + \frac{1}{2}4 = 2$. Since his utility from the proposed mix is at least as high as his utility from the other option (U), the proposed mix is a mixed equilibrium.

Possibility 4: 1 mixes on U, C and D, 2 mixes on M and R Let p be the probability of U, r be the probability on C, and q be the probability on M.

For this to work, P1 would need to be indifferent between all three of his choices. Note that this is equivalent to saying he is indifferent between U and C and between C and D.

To be indifferent between U and C, in possibility 1, we found that for that to be true, P2 must be weighting M and R each with probability $\frac{1}{2}$.

To be indifferent between C and D, in possibility 3, we found that for that to be true, P2 must be weighting M and R each with probability $\frac{1}{2}$.

Since both indifferences require a 50/50 weighting on M and R, P1 is indeed indifferent between all three of his actions is P2 os weighting 50/50!

The next step is to see if P2 can be made indifferent from P1 playing some mix over his three actions.

$$\text{To make P2 indiff:} \qquad 2p + 4r + 3(1-p-r) = 3p + 4(1-p-r) \qquad \Leftrightarrow r = \frac{1}{5}$$

Note that the only necessity here is that $r = \frac{1}{5}$, i.e. that P1 puts weight $\frac{1}{5}$ on C. Therefore, we have a continuum of mixed equilibrium $\left((\gamma, \frac{1}{5}, \frac{4}{5} - \gamma), (0, \frac{1}{2}, \frac{1}{2})\right)$ for all $\gamma \in [0, \frac{4}{5}]$. Note that this continuum actually subsumes the equilibria we found in possibilities 1 and 3! Therefore, the full set of mixed equilibria can be written as CM, UR and the continuum we just found above.

2. Consider the following social problem. A pedestrian is hit by a car and lies injured on the road. There are n people in the vicinity of the accident. The injured pedestrian requires immediate medical attention, which will be forthcoming if at least one of the n people calls for help. Simultaneously and independently each of the n bystanders decides whether or not to call for help (by dialing 911). Each bystander obtains v units of utility if anyone calls for help. Those who call for help pay a personal cost of c. That is, if person i calls for help, then he obtains the payoff $v - c$; if person i does not call but at least one other person calls, then person i gets v; finally, if none of the n people calls for help, then person i obtains zero. Assume $v > c$.

    (a) Describe the NE in pure strategies.

    (b) Find the symmetric mixed equilibria (i.e. all players do the same). In your analysis, let p be the probability that a person doesn't call for help.

    (c) Compute the probability that at least one person calls for help in the mixed equilibrium. Comment on how this depends on n and whether the result is intuitive?

    Solution:

    For (a), any profile in which exactly one person calls is a NE. To see this, note that those not calling prefer not to call because their call would cost them c and give them no benefit (since somebody else is calling). For the person calling, not calling would save them c but sacrifice $v > c$ since nobody else is calling. Therefore, he has no incentive to deviate.

    For (b), for any player to mix, he must be indifferent between calling and not calling, so his expected utility must be the same in either case. There are basically two relevant states of the world to consider for the agent, the state in which nobody else has called (denote it N for No) which occurs with probability $p^{(n-1)}$ and the state in which somebody else has called (denote it Y for Yes) (how many doesn't matter), which occurs with probability $1-p^{(n-1)}$. Therefore, to be indifferent between calling and not calling

$$\begin{aligned} \bar{u}_i(\text{calling}) &= \bar{u}_i(\text{not calling}) \\ P(N)(v-c) + P(Y)(v-c) &= P(N)(0) + P(Y)(v) \\ v - c &= p^{(n-1)} \cdot 0 + (1 - p^{(n-1)})v \\ p &= \left(\frac{c}{v}\right)^{\frac{1}{n-1}} \end{aligned}$$

Therefore, if each calls with probability $1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}}$, then each is indifferent between calling and not and this is a mixed equilibrium. Note that this p is increasing in n because $c/v < 1$. This makes sense - the more people there are, the less likely an individual chooses to call in this eqm. But there are more individuals, so hopefully the person still gets the help they need.

For (c), note that the probability that nobody calls, since they're all randomizing independently, is just the product of the probabilities that each person doesn't call. That is, the probability that nobody calls is

$$P(\text{Nobody Calls}) = \left(\left(\frac{c}{v}\right)^{\frac{1}{n-1}}\right)^n = \left(\frac{c}{v}\right)^{\frac{n}{n-1}}$$

If you differentiate this with respect to n, you'll see it is increasing. Therefore, the more people there are, the lower the likelihood that *anybody* calls. You can even think of what happens as we take the limit of people as n goes to infinity.

$$\lim_{n \to \infty} \left(\frac{c}{v}\right)^{\frac{n}{n-1}} = \frac{c}{v}$$

This is only true if $c < v$. In particular, if c is very close v, then as we get more and more people, the probability that nobody calls goes to nearly one.

This is counter-intuitive (and perverse) in the sense that the more people see the accident, the less likely it is to be called in.

3. Consider the following 3-player game where P1 chooses the row, P2 chooses the column, and P3 chooses the bimatrix (A or B). P1's payoff is listed first, P2's is listed second, and P3's is listed third. Find the pure-strategy NE.

A	L	R
U	$2, 0, 2$	$1, 1, 1$
D	$1, 1, 1$	$0, 2, 2$

B	L	R
U	$0, 2, 2$	$1, 1, 1$
D	$1, 1, 1$	$2, 0, 2$

Solution: Just underline best responses for each player given each of the four profiles of opponents' play. You'll find URA and DLB are NE.

4. The Allies have one bomber plane which they can use to strike one of three possible targets: a dam (D), an airstrip (A) and a tank (T). The values of those targets are $v_D = 4, v_A = 3, v_T = 2$. The Axis has one anti-aircraft gun, and can choose to defend only one of the three targets. The bomber destroys the target if it is undefended, and does no damage if it is defended. The Allies get utility equal to the value of the object they destroy, if they destroy an object. The Axis gets utility equal to negative the value of an object destroyed, if an object is destroyed. As an example, listing the Allies choice first, $u_{Allies}(D, T) = 4, u_{Axis}(D, T) = -4$. $u_{Axis}(D, D) = u_{Allies}(D, D) = 0$. The Allies and Axis make their decisions simultaneously. Find a mixed equilibrium in which both the Axis and Allies mix across all three of their actions.

Solution:

For the Allies to mix across their actions, they must be indifferent, i.e.

$$\bar{u}_{Allies}(D) = \bar{u}_{Allies}(A) \quad \Leftrightarrow \quad (\alpha_{Axis}(A) + \alpha_{Axis}(T))4 = (\alpha_{Axis}(D) + \alpha_{Axis}(T))3$$
$$\bar{u}_{Allies}(A) = \bar{u}_{Allies}(T) \quad \Leftrightarrow \quad (\alpha_{Axis}(D) + \alpha_{Axis}(T))3 = (\alpha_{Axis}(D) + \alpha_{Axis}(A))2$$

and $\alpha_{Axis}(D) + \alpha_{Axis}(A) + \alpha_{Axis}(T) = 1$. Solving this system of equations yields $\alpha_2^* = \left(\frac{7}{13}, \frac{5}{13}, \frac{1}{13}\right)$

For the Axis to mix across their actions, they must be indifferent, i.e.

$$\begin{aligned}
\bar{u}_{Axis}(D) = \bar{u}_{Axis}(A) \quad &\Leftrightarrow \quad \alpha_{Allies}(A)(-3) + \alpha_{Allies}(T)(-2) = \alpha_{Allies}(D)(-4) + \alpha_{Allies}(T)(-2) \\
\bar{u}_{Axis}(A) = \bar{u}_{Axis}(T) \quad &\Leftrightarrow \quad \alpha_{Allies}(D)(-4) + \alpha_{Allies}(T)(-2) = \alpha_{Allies}(D)(-4) + \alpha_{Allies}(A)(-3)
\end{aligned}$$

and $\alpha_{Allies}(D) + \alpha_{Allies}(A) + \alpha_{Allies}(T) = 1$. Solving this system of equations yields $\alpha_1^* = \left(\frac{3}{13}, \frac{4}{13}, \frac{6}{13}\right)$

(From lecture notes pages 80-81)

A general Algorithm (semi-formal). In two-player games with finite strategy sets, one can find all (pure and mixed) Nash equilibria using the following procedure:

(a) Determine each player's rationalizable strategies $\rho_i^\infty(A)$.

(b) Pick a player $i \in \{1, 2\}$ (for instance, the one with fewer strategies). For each non empty subset $C_i \subseteq \rho_i^\infty(A)$, find all Nash equilibria with $Supp(\alpha_i) = C_i$.

Why is this a good procedure? If α is a Nash equilibrium with $Supp(\alpha_1) = C_1$, then all strategies in C_1 must be optimal. This has implications for α_2. But since α_2 must also be optimal, there are in turn implications for α_1. In the end, one either finds one or more equilibria or reaches a contradiction.

A General Algorithm (formal). Let $[u_i^{k\ell}]$ be the payoff matrix of player i, with generic entry $u_i^{k\ell}$. Player 1 chooses the rows (indexed by k) and player 2 the columns (indexed by ℓ).

Step 1: Eliminate all iteratively dominated actions (by Theorem 5 and Corollary 2 such actions are played with zero probability in equilibrium). The order of elimination is irrelevant.

Step 2: For any pair of non-empty subsets $A_1^* \subseteq A_1$ and $A_2^* \subseteq A_2$, compute the set of mixed equilibria, (α_1, α_2), such that $\text{Supp}\alpha_1 = A_1^*$ and $\text{Supp}\alpha_2 = A_2^*$. This set, which could be empty, is computed as follows (we consider the non trivial case in which the sets contain at least two actions). To simplify the notation, assume that $A_1^* = \{1, ..., m_1^*\}$ and $A_2^* = \{1, ..., m_2^*\}$ and denote by α_i^m the generic probability that action m of player i is played. Solve the following systems of linear equations and inequalities with unknown $\alpha_1 = (\alpha_1^1, ..., \alpha_1^{m_1^*}) \in \Delta(A_1^*)$ and $\alpha_2 = (\alpha_2, ..., \alpha_2^{m_2^*}) \in \Delta(A_2^*)$:

$$\begin{aligned} \sum_{k=1}^{m_1^*} u_2^{k\ell}\alpha_1^k &= \sum_{k=1}^{m_1^*} u_2^{k1}\alpha_1^k,\ \ell = 2, ..., m_2^*, \\ \sum_{k=1}^{m_1^*} u_2^{k\ell}\alpha_1^k &\leq \sum_{k=1}^{m_1^*} u_2^{k1}\alpha_1^k,\ \ell = m_2^* + 1, ..., |A_2^*|; \end{aligned} \tag{1}$$

$$\begin{aligned} \sum_{\ell=1}^{m_2^*} u_1^{k\ell}\alpha_2^\ell &= \sum_{\ell=1}^{m_2^*} u_1^{1\ell}\alpha_2^\ell,\ k = 2, ..., m_1^*, \\ \sum_{\ell=1}^{m_2^*} u_1^{k\ell}\alpha_2^\ell &\leq \sum_{\ell=1}^{m_2^*} u_1^{1\ell}\alpha_2^\ell,\ k = m_1^* + 1, ..., |A_1^*|. \end{aligned} \tag{2}$$

The subset of equations in (1) determines the set of mixed actions of player 1 that make player 2 indifferent between the actions in subset A_2^*. The subset of inequalities determines the set of mixed actions of player 1 that make the action $a_2 = 1$ (and so *all the actions in* A_2^*) weakly preferred to the actions that do not belong to A_2^*. For any α_1 that satisfies the system (1), player 2 has no incentive to "deviate" from a mixed action with support A_2^*. Similar considerations hold for system (2). Such system determines the set of mixed actions of player 2 that make player 1 indifferent among all the actions in A_1^* and at the same time make such actions weakly preferred to all the others. Therefore, *the indifference conditions for player 1 determine the equilibrium randomization(s) of player 2, the indifference conditions for player 2 determine the equilibrium randomization(s) of player 1.*