ECON 521, Discussion Section 5

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1. Suppose there are three candidates in an election and the set of possible positions they can take is $\{0, 1, 2, \ldots, l\}$, where l is even. Each voter has his own position, and will vote for whichever candidate is closest to his position. If two candidates have the same position, voters closest to that position will split evenly across the two candidates. Each candidate has the option of staying out of the race, which she regards as better than losing and worse than tying for first place. Show that if less than one third of the citizens' favorite positions are equal to the median favorite position (m), then the game has no Nash equilibrium. Argue as follows. First, show that the game has no Nash equilibrium in which a single candidate enters the race. Second, show that in any Nash equilibrium in which more than one candidate enters, all candidates that enter tie for first place. Third, show that there is no Nash equilibrium in which two candidates enter the race. Fourth, show that there is no Nash equilibrium in which all three candidates enter the race and choose the same position. Finally, show that there is no Nash equilibrium in which all three candidates enter the race and do not all choose the same position.

   Solution:

   First: Suppose there exists a Nash equilibrium (NE) in which exactly one candidate enters. Then, one of the candidates not entering could, at the very least, enter and choose the same position as the first candidate, therefore tying for first. Since tying for first is better than staying out, one of the out candidate would have incentive to do so. Therefore, there can be no NE in which exactly one candidate enters.

   Second: Suppose there exists a NE in which multiple candidates enter and they do not tie. Then one loses. But then the loser would prefer to not enter. Therefore, in any NE in which more than one candidate enters, they must all tie for first.

   Third: Suppose there exists a NE in which exactly two candidates enter. By the point above, they must be tying for first. For this to be true, they must be getting the same number of votes. If one is positioned away from m, the other could win the election by positioning on m, therefore for the to be tying, they must both be on m. Given this, a thrid player could also tie by entering and also positioning on m. Since tying is preferred to not entering, the out candidate would prefer to do so. Therefore, there can be no NE in which exactly two candidates enter.

   Fourth: Suppose there exists a NE in which all three candidates enter and choose the same position. If that position is not m, then any candidate could win outright by choosing m. If that position is m, then any candidate could win outright by choosing either $m-1$ or $m+1$ since, because less than a third of voters are at the median position, there is strictly more than one third of voters above and strictly more than one third below. By capturing that percentage, and having the other two splitting the remaining votes (less than 2/3), the candidate choosing $m-1$ or $m+1$ would win outright.

   Finally: Suppose there exists a NE in which all three candidates enter and choose different positions. By the second point, they must all tie. If they are tying on different positions, they must be evenly spread across the distribution of voters, i.e. $a_1 < a_2 = m < a_3$. Since there is less than one third of voters at m and player 2 is tying for first, there must exist a position b_1 between a_1 and a_2 and a position b_3 between a_2 and a_3. But then, candidate one could win outright by deviating to b_1.

Similarly, candidate three could win outright be deviating to b_3. Therefore, there can be no NE in which all three candidates enter and choose different positions.

Since the points above describe all possible NE (and rule them out), we can conclude that there exists no NE in this game.

2. Consider the following n-player game. Simultaneously and independently, the players each select either X, Y, or Z. The payoffs are defined as follows. Each player who selects X obtains a payoff equal to γ, where γ is the number of players who select Z. Each player who selects Y obtains a payoff of 2α where α is the number of players who select X. Each player who selects Z obtains a payoff of 3β, where β is the number of players who select Y. Note that $\alpha + \beta + \gamma = n$.

   (a) Suppose $n = 2$. Represent this game in the normal form by drawing the appropriate matrix.

   (b) In the case of $n = 2$, does this game have a NE in pure strategies? Does it have a mixed equilibrium? In either case, find the equilibrium.

   (c) Suppose $n = 11$. Does this game have a NE? If so, describe the equilibrium.

   Solution:

   For (a), the game should look as follows:

G :

	X	Y	Z
X	0,0	0,2	1,0
Y	2,0	0,0	0,3
Z	0,1	3,0	0,0

For (b), you should find that it has no pure-strategy NE. In terms of a mixed equilibrium, there are several possibilities.

Is there a mixed equilibrium where at least one player mixes over exactly two strategies? If one player mixed over X and Y, for instance, then the other player would never want to play X (because X is only good against Z). But given that, the original player wouldn't put weight on Y, which only good against X. You can repeat this reasoning to rule out any mixed equilibrium where either player puts positive probability on only two actions.

Therefore, we need to look for a mixed equilibrium where both players put positive weight on all three actions. Since the game is symmetric, we can just find probabilities that one player must play X, Y and Z such that the other is indifferent between X, Y, and Z. In particular

$$\bar{u}_1(X, \alpha_2) = \bar{u}_1(Y, \alpha_2)$$
$$\alpha_2(Z) \cdot 1 = \alpha_2(X) \cdot 2$$
$$\bar{u}_1(Y, \alpha_2) = \bar{u}_1(Z, \alpha_2)$$
$$\alpha_2(X) \cdot 2 = \alpha_2(Y) \cdot 3$$

Combining the system of equations above with $\alpha_2(X) + \alpha_2(Y) + \alpha_2(Z) = 1$ gives us that $\alpha_2 = (\frac{3}{11}, \frac{2}{11}, \frac{6}{11})$. Because of the symmetry of the game, $\alpha_2 = (\frac{3}{11}, \frac{2}{11}, \frac{6}{11})$. Then (α_1, α_2) is a mixed equilibrium.

For (c), this game has a NE in which exactly three players play X, exactly 2 play Y, and 6 play Z. In this case, all players get the same payoff of 6, and none could increase his payoff by deviating to another action.

3. Argue that if iterated deletion of strictly dominated strategies eliminates all but one outcome, then (a) that outcome is a Nash equilibrium and (b) there cannot be any other Nash equilibria. (Hint: Use the relationships between rationalizability and iterated deletion of strictly dominated strategies, and Corollary 2 in Chapter 5 of the lecture notes.)

   Solution:

   For (a), if iterated deletion of strictly dominated strategies eliminates all but one outcome, then only the actions that lead to that outcome are rationalizable (see Theorem 5 in Chapter 4 of LN). Since existence of a mixed equilibrium is guaranteed, and Corollary 4 tells us that mixed equilibria cannot have weight on actions that are not rationalizable, we can conclude that the mixed equilibrium must be that unique rationalizable outcome. Since it places all weight on one action for each player, it is also a NE.

   For (b), we know that mixed equilibria cannot have positive weight on actions that are not rationalizable (again, Corollary 4). Since NE are a subset of mixed equilibria, the same must be true for NE. Since there are no other rationalizable strategies, there can be other NE.

4. Suppose you know the following about a particular two-player game: $A_1 = \{A, B, C\}$, $A_2 = \{X, Y, Z\}$, $u_1(A, X) = 6$, $u_1(A, Y) = 0$, and $u_1(A, Z) = 0$. In addition, suppose you know that the game has a mixed equilibrium in which (a) the players select each of their actions with positive probability, (b) player 1's expected payoff in equilibrium is 4, and (c) player 2's expected payoff in equilibrium is 6. Do you have enough information to calculate the probability that player 2 selects X in equilibrium? If so, what is the probability?

   Solution:

   Since P1 has an expected utility of 4 and he is indifferent between his three actions (he must be to be mixing), he must have an expected utility of four from each action. Given that $u_1(A, Y) = 0$, and $u_1(A, Z) = 0$, the only way to have P1 get expected utility of 4 from playing A is if P2 is putting probability of $\frac{2}{3}$ on X. Telling you that P2 selects each of her actions with positive probability and that player 2's expected payoff in equilibrium was 6 was just to mess with you – you didn't need to know that.

5. Antonio's Spaghetti Factory Ltd. has three creditors. Each creditor is owed d dollars, and the value of the firm is V. The firm is insolvent in that $3d > V$. If just one creditor collects, the firm can pay the debt. If two or more creditors attempt to collect, the firm goes bankrupt (i.e. the firms fail to collect, but still have to pay the collection cost). If one or more creditors collects, the firms value is reduced to v. The costs of collection for each firm is c dollars. The payoff matrix is therefore as follows, where the cell in the i^{th} row and the j^{th} column represents the payoff to a firm who chooses the i^{th} and j other firms chooses to collect.

# others collecting	0	1	2
collect	$d - c$	$v/3 - c$	$v/3 - c$
refrain	$V/3$	$(v - d)/2$	$v/3$

Supposing $d - c > \frac{V}{3}$ and $\frac{v-d}{2} < \frac{v}{3} - c$ what is the NE? Would the three creditors support a law in which they were forbidden to collect from Antonio's Spaghetti Factory Ltd.?

Solution:

Since the best response to nobody else collecting is to collect, and the best response to one other person collecting is to collect, but the best response to two people collecting is to not collect, the NE is where exactly two people collect. Since the payoffs in this NE are $v/3 - c$ and $v/3$, which are both less than $V/3$, they would support a law that forbid them all from collecting.

6. Consider the following three player game in which P1 chooses the row, P2 chooses the column, and P3 chooses the matrix.

	L	R
T	$0, 0, 3$	$0, 0, 0$
B	$1, 0, 0$	$0, 0, 0$

X

	L	R
T	$2, 2, 2$	$0, 0, 0$
B	$0, 0, 0$	$2, 2, 2$

Y

	L	R
T	$0, 0, 0$	$0, 0, 0$
B	$0, 1, 0$	$0, 0, 3$

Z

(a) Find the pure strategy NE and list the payoffs for each.

(b) Show that there is a correlated equilibrium in which P3 chooses Y and P1 and P2 play (T, L) and (B, R) with equal probabilities.

(c) Explain the sense in which P3 prefers not to have the information that P1 and P2 use to coordinate their actions.

Solution:

Underlining the best responses for each profile of opponent actions, for each player, gives us the NE of TRX, BLX, TRZ, and BLZ. The payoffs from these NE are $(0, 0, 0), (1, 0, 0), (0, 0, 0), (0, 1, 0)$ respectively. Notice that P3's payoff is zero in each case.

To check that the proposition in (b) is in fact a correlated equilibrium, we have to check that each player is best responding to the others. For P1, given that P2 is playing L and P3 is playing Y, T is indeed a best response, as is B when P2 is playing R and P3 is playing Y. For P2, given that P1 is playing T and P3 is playing Y, L is his best response, as is R when P1 is playing B and P3 is playing Y.

As for for P3, is Y his best response to the the 50/50 mix on (T, L) and (B, R)? By doing that, he gets 2. If he deviated to X or Z, he would get expected utility of 1.5. Therefore it his best response to play Y and this is a correlated equilibrium.

If P3 could observe the coin flip for P1 and P2, then he would best respond to TL with X and get 3 and best respond to BR with Z and get 3. However, if he were doing that, P1 and P2 wouldn't want to mix over TL and BR in that way, and presumably one of the NE would result (or maybe a mixed) and P3 would get zero. In other words, by not observing the coin toss, P3 avoids the temptation to screw his opponents. Knowing that P3 can't screw them (because he doesn't observe the toss), P1 and P2 are willing to coordinate on something that works pretty well for everyone in that it gives each player more utility than he would get from any of the NE.