ECON 521, Discussion Section 6

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1. Consider the following game (basically Chicken/Hawk-Dove)

G :

	d	c
D	0,0	7,2
C	2,7	6,6

(a) Find all Nash and Mixed equilibria.

Solution: NE are $(C, d), (D, c)$ and ME is $\left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$. Expected payoffs are $(2, 7), (7, 2)$ and $\left(\frac{14}{3}, \frac{14}{3}\right)$ respectively. Note $\frac{1}{9}0 + \frac{2}{9}2 + \frac{2}{9}7 + \frac{4}{9}6 = \frac{14}{3}$. That's where that comes from.

(b) Suppose you have access to a fair coin and can instruct each player on what action to play conditionally on the outcome of a coin-toss. Players are aware of the method by which you are instructing them, but observe only your instruction to them, not the outcome of the randomization. Construct a correlated equilibrium in which the two players have the same expected payoff and compare the expected payoffs to those of the mixed equilibrium you found in (a).

Solution: In solving this problem, I'm going to refer to myself as a fictitious mediator that can run the randomization and then send messages to the two players. So when I say going forward that I am going to *instruct* a player to do something, I mean that I will send that player a message (contingent on the random outcome), perhaps $t_i =$ Play C and then his instruction will be $\delta_i(\text{Play } C) = C$.

If heads, I instruct them to play (C, d). If tails, I instruct them to play (D, c). Expected payoffs from this is obviously just $\frac{7+2}{2} = \frac{9}{2}$, since they're each just getting 7 and 2 with equal probability. Note something counterintuitive here, $\frac{9}{2} < \frac{14}{3}$, so this correlated equilibrium is actually worse than the mixed equilibrium! But it is still a correlated equilibrium, as neither would have incentive to deviate.

(c) Now suppose you have access to a three-sided die instead of the coin. Can you find a correlated equilibrium that gives both players more utility than both the mixed in (a) and the correlated in (b)?

Solution: If a 1 is rolled, I instruct P1 to play C and P2 to play d. If a 2 is rolled, I instruct P1 to play D and P2 to play c. If a 3 is rolled, I instruct both to play C/c. Remember, they know what I have instructed them (and how I'm deciding what to instruct them), but they don't see the die roll itself. When instructed to play D/d, they know that I have instructed the opponent to play C/c. Since D/d is the best response to C/c, they have no incentive to deviate. When I instruct them to play C/c, they know that there's a 50% chance I've instructed their opponent to play C/c and a 50% chance I've instructed their opponent to play D/d. Given this, I need them to actually want to follow my instruction.

$$\bar{u}(C/c) \geq \bar{u}(D/d)$$
$$\frac{1}{2}2 + \frac{1}{2}6 \geq \frac{1}{2}0 + \frac{1}{2}7$$
$$4 \geq 3.5 \quad \checkmark$$

As for the expected utility in this, they end up with $\frac{1}{3}2 + \frac{1}{3}7 + \frac{1}{3}6 = \frac{15}{3}$. Great news, that's our best expected payoff so far!

(d) Finally, replace that three-sided die with a random-number generator that draws uniformly from $[0, 1]$. Find the best possible correlated eqm, i.e. that which maximizes expected payoffs. Restrict your attention to symmetric equilibria, i.e. both players have the same expected payoff.

Solution: This is a little harder, and perhaps a little beyond the scope of this course. Hopefully, you've got the gist from the parts above that we basically want to use this private correlating device to try to trick them into (C, c) as much as possible. When instructed to play D/d, that player knows the other has been instructed to play C/c. Since D/d is the best response to C/c, there's no incentive to deviate. It's when we instruct them to play C/c that they might think of deviating. If we instruct both to play C/c too often (i.e. with too much weight), each will have incentive play D/d when instructed to play C/c because he will believe that there is sufficiently high probability that the other has been instructed to play C/c, to which D/d is the best response. So, the goal is to put as much weight on (C, c) as possible subject to the constraint that the players are still willing to play C/c when we instruct them to do so. Therefore, let's instruct them both to play C/c with probability p. Then we'll instruct P1 to play C and P2 to play d with probability $\frac{1-p}{2}$ and P1 to play D and P2 to play c with probability $\frac{1-p}{2}$ also.

Given that you are instructed to play C/c then, the probability that your opponent was also instructed to play C/c is (by Bayes rule)

$$P(t_{-i} = C/c | t_i = C/c) = \frac{P(t_{-i} = C/c \cap t_i = C/c)}{P(t_i = C/c)} = \frac{p}{p + \frac{1-p}{2}} = \frac{2p}{p+1}$$

Then, the goal is

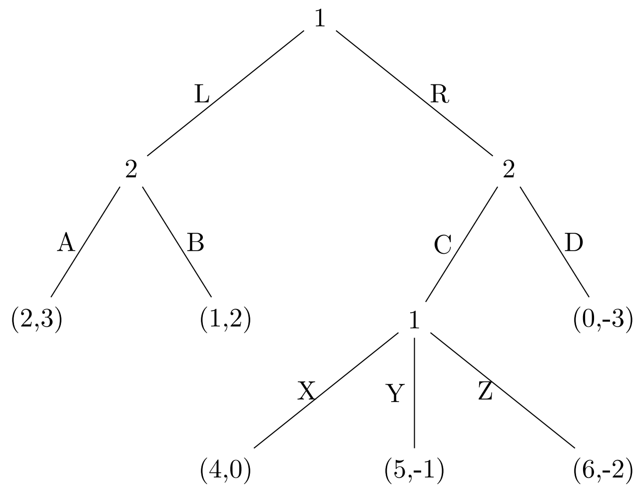
$$\max_p 6p + \frac{1-p}{2}7 + \frac{1-p}{2}2$$
$$\text{s.t. } \bar{u}(C/c | t_i = C/c) \geq \bar{u}(D/d | t_i = C/c)$$
$$\text{that is } \frac{2p}{p+1}6 + (1 - \frac{2p}{p+1})2 \geq \frac{2p}{p+1}7 + (1 - \frac{2p}{p+1})0$$
$$\text{that is } p \leq \frac{1}{2}$$

Looking at the maximization problem, it is just increasing in p, so we want to make p as high as possible. Therefore, the maximum is where the constraint binds. And solving the constraint above with equality gives us $p = \frac{1}{2}$. Therefore, with probability 1/2 we will instruct them both to play C/c. Then we will instruct (C, d) and (D, c) each with probability 1/4, and this is the correlated equilibriun that maximizes total utility.

To implement this with the random number generator, we can instruct them both to play C/c if the outcome is in $[0, 1/2)$. We can instruct (C, d) if it's in $[1/2, 3/4)$ and (D, c) if it's in $[3/4, 1]$.

2. Consider the following game, Γ:

Γ :



(a) Find H, Z, $(\bar{A}_i, A_i(\cdot))_{i\in I}$.

> **Solution:** Let's define ϕ as the initial history, then the partial histories are $H = \{\phi, L, R, RC\}$ and the terminal histories are $Z = \{LA, LB, RD, RCX, RCY, RCZ\}$. The feasible actions are $\bar{A}_1 = \{L, R, X, Y, Z\}$ and $\bar{A}_2 = \{A, B, C, D\}$. $A_1(\phi) = \{L, R\}$. $A_1(\phi) = \{L, R\}$ $A_2(L) = \{A, B\}$. $A_2(R) = \{C, D\}$. $A_1(RC) = \{X, Y, Z\}$.

(b) Find the normal form, reduced normal form, and the NE.

> **Solution:**

Normal Form :

	AC	AD	BC	BD
LX	2,3	2,3	1,2	1,2
LY	2,3	2,3	1,2	1,2
LZ	2,3	2,3	1,2	1,2
RX	4,0	0,-3	4,0	0,-3
RY	5,-1	0,-3	5,-1	0,-3
RZ	6,-2	0,-3	6,-2	0,-3

Reduced Normal Form :

	AC	AD	BC	BD
L	2,3	2,3	1,2	1,2
RX	4,0	0,-3	4,0	0,-3
RY	5,-1	0,-3	5,-1	0,-3
RZ	6,-2	0,-3	6,-2	0,-3

> The NE are just (L, AD), (RZ, AC) and (RZ, BC).

(c) Find the backward induction solution.

> **Solution:** The backward induction solution is RCZ.

(d) Are there any NE which seem to rely on threats that are not credible?

> **Solution:** The NE involving L and B rely on non-credible threats by P2 to play D and B respectively.