ECON 521, Discussion Section 9

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1. Evaluate the present value of each of the following infinite sequences given discount rate δ.

   (a) $(1, 1, 1, \ldots)$

   Solution: The value of this sequence is $v(s) = 1 + \delta + \delta^2 + \delta^3 + \ldots$. Note that

$$\begin{aligned} 1 + \delta + \delta^2 + \delta^3 \ldots &= 1 + \delta(1 + \delta + \delta^2 + \delta^3 + \ldots) \\ v(s) &= 1 + \delta(v(s) \\ v(s) &= \frac{1}{1-\delta} \end{aligned}$$

   (b) $(x, x, x, \ldots)$

   Solution: This is just the same as the one directly above except x insteado of 1 in each period, hence $v(s) = \frac{x}{1-\delta}$.

   (c) $(0, 0, x, x, x, \ldots)$

   Solution: This is the same as the one directly above except it starts two periods later. Therefore, we discount it twice, so $v(s) = \delta^2 \frac{x}{1-\delta}$

   (d) $(2, 0, 2, 0 \ldots)$

   Solution: Here $v(s) = 2 + 2\delta^2 + 2\delta^4 + 2\delta^6 + \ldots$. But

$$\begin{aligned} 2 + 2\delta^2 + 2\delta^4 + 2\delta^6 &= 2 + \delta^2(2 + 2\delta^2 + 2\delta^4 + 2\delta^6) \\ v(s) &= 2 + \delta^2 v(s) \\ v(s) &= \frac{2}{1-\delta^2} \end{aligned}$$

   (e) $(0, 2, 0, 2 \ldots)$

   Solution: This is the same as the one directly above except delayed by one period. Hence $v(s) = \delta \frac{2}{1-\delta^2}$.

   (f) $(1, 2, 3, 1, 2, 3, \ldots)$

   Solution: Treat this as three separate sequences, one of 1's every third period starting in period 1, one of 2's every third period starting in period 2, and one of 3's every third period starting in period 3. Then the first is worth $v(s_1) = \frac{1}{1-\delta^3}$. The second is worth $v(s_2) = \delta \frac{2}{1-\delta^3}$ and the third is worth $v(s_3) = \delta^2 \frac{3}{1-\delta^3}$. Then for the whole sequence

$$v(s) = v(s_1) + v(s_2) + v(s_3) = \frac{1}{1-\delta^3} + \delta \frac{2}{1-\delta^3} + \delta^2 \frac{3}{1-\delta^3}$$

2. Suppose you are considering two possible infinite sequences of payoffs, where d is the sequence resulting from deviating and b is the sequence resulting from behaving:

$$d = (w, x, x, x, \ldots) \qquad\qquad b = (y, z, z, z, \ldots)$$

   Why is it appropriate to say that the deviation is profitable if and only if $(1-\delta)w + \delta x > (1-\delta)y + \delta z$?

Solution: This one is nice and simple but I wanted to show it to you because you'll often see it represented in this form. The present value of the payoff sequence d is $v(d) = w + \delta\frac{x}{1-\delta}$ and the present value of payoff sequence b is $v(b) = y + \delta\frac{z}{1-\delta}$. Therefore, the deviation is profitable if and only if

$$w + \delta\frac{x}{1-\delta} > y + \delta\frac{z}{1-\delta}$$

Simply multiply both sides by $(1-\delta)$ and you get the original statement:

$$(1-\delta)w + \delta x > (1-\delta)y + \delta z$$

Again - this is a simple point, but you'll often see it written this way, so I wanted to make sure that everybody knew where it was coming from. When you're multiplying things by $(1-\delta)$, you're essentially considering the average flow value you get in each period, so these are sometimes called *average discounted payoffs.*

3. Consider the following prisoner's dilemma game (slightly different payoffs from what he had in class, but it's this way in some textbooks):

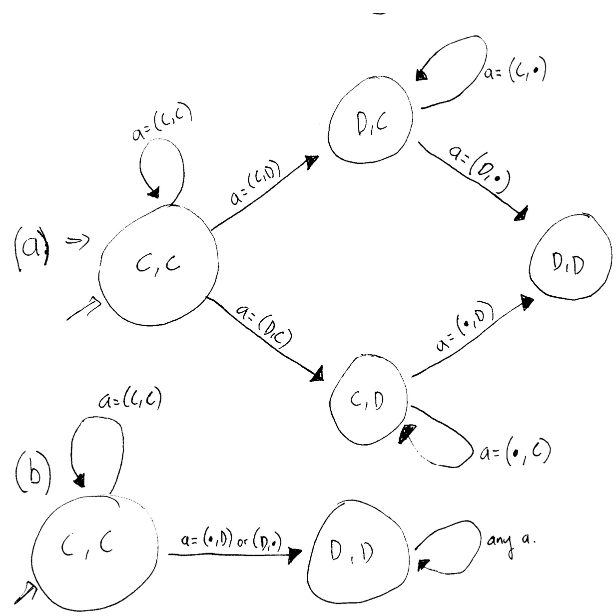
	C	D
C	2,2	0,3
D	3,0	1,1

You could interpret the grim trigger strategy in two ways. For each of the following interpretations, represent the strategy profile as an autonoma and find whether or not it is a SPE (and if so, for what range of δ).

(a) Each player plays C so long as the other played C in the previous round. If a player plays D, **the other player** plays D in all subsequent rounds.

(b) Each player plays C so long as the other played C in the previous round. If a player plays D, **both players** play D in all subsequent rounds.

Solution: Here are the autonoma representations:



First let's check if (b) is a SPE. For (b), obviously there's no incentive to deviate in the state on the right because each should play D, their actions don't affect the future state, and D is the best response to D. Is there an incentive to deviate from that? Well, from playing C, the payoff is $v(C) = \frac{2}{1-\delta}$. From deviating to play D, the payoff is $v(D) = 3 + \frac{1}{1-\delta}$. Therefore, this is a SPE if and only if

$$\begin{aligned} v(C) &\geq v(D) \\ \frac{2}{1-\delta} &\geq 3 + \frac{1}{1-\delta} \\ \delta &\geq \frac{1}{2} \end{aligned}$$

Now let's check if (a) is an SPE. To conclude that it isn't, we need only find that there exists one state in which there is an incentive to deviate. Let's look at (C, D) – you could also look at (D, C). In (C, D), player 1 should play C. But since player 2 should play D, for player 1, $v(C) = 0 + \delta\frac{1}{1-\delta}$ and $v(D) = 1 + \delta\frac{1}{1-\delta}$. But then $v(D) > v(C)$ for all $\delta \in (0, 1)$, and there's an incentive to deviate. Hence, the interpretation of grim-trigger in (a) is not an SPE. Intuitively, what's going on is that once a player cheats, he knows that the other player will play D in every subsequent period. Therefore, his best response is to play D in every period also. That's why the correct (or SPE) grim-trigger is that in which they both punish each other as soon as either deviates, as in (b).

4. Roughly speaking, the folk theorem shown in the lecture applied to the prisoner's dilemma says that no matter how minor the stage-game punishment NE (D, D), and no matter how large the payoff from deviating is, there exists a δ such that that punishment is severe enough to incentivize players to play (C, C) in every period. That is, even if the prisoner's dilemma is as follows,

	C	D
C	2,2	0,999999
D	999999,0	2-ϵ, 2-ϵ

where ϵ is an arbitrarily small positive number, there still exists a δ that's very close to 1 such that grim-trigger (in which the outcome is play they (C, C) in every period) is an SPE.

Make a similar argument in the other direction. That is, in the following game

	C	D
C	2,2	x-1,3
D	3,x-1	x,x

where $x \in \mathbb{R}$, show that for any $\delta > 0$, there exists an x^* such that for all $x < x^*$, there exists a grim trigger SPE in which the outcome is that each player plays C in every period. In words, you're trying to argue that even if a person is very myopic, i.e. barely cares about the future at all, you can still come up with a future punishment severe enough such that they play C. Of course, if $\delta = 0$, this isn't true. Comment on what happens to x^* as $\delta \to 0$ and as $\delta \to 1$.

Solution: Let's again suppose they play a grim-trigger strategy in which as soon as either player plays D, they both play D forever on. By the same argument as before, when they're both instructed to play (D, D), there can be no profitable deviation, because the state remains the same regardless of their actions and D is the best response to D. So we just need to check the state in which they're both told to play

C. Then

$$\begin{aligned} V(C) &\geq V(D) \\ \frac{2}{1-\delta} &\geq 3 + \delta \frac{x}{1-\delta} \\ x &\leq \frac{3\delta - 1}{\delta} \end{aligned}$$

Therefore, for any δ, there exists an $x^* = \frac{3\delta-1}{\delta}$ such that for all $x < x^*$, the grim trigger strategy profile is a SPE. Note that as $\delta \to 1$, $x^* \to 2$. So as people get more and more patient, the punishment NE can be just neglibibly worse than the profile we're trying to sustain. On the other hand as $\delta \to 0$, $x \to -\infty$. So if people are very myopic, the punishment must be extremely severe to incentivize (C, C).

5. Suppose that you are playing the infinitely-repeated bertrand duopoly - that is, two firms compete on price period after period indefinitely. Each firm's unit cost is constant, equal to c. Denote the total demand for the good at the price p by $D(p)$. Let $\pi(p) = (p - c)D(p)$ for very price p, and assume that D is such that the function π is continuous and has a single maximizer, denoted by p^m, i.e. the monopoly price.

   (a) Let s_i be the strategy of firm i in the infinitely-repeated game of this strategic game that charges p^m in the first period and subsequently as long as the other firm continues to charge p^m and punishes any deviation from p^m by the other firm by choosing the price c for k periods, then reverting to p^m. Given any value of δ, for what values of k is the strategy pair (s_1, s_2) a NE?

   (b) Let s_i be the following strategy for firm i in the infinitely repeated game:

   - in the first period charge the price p^m
   - in every subsequent period, charge the lowest of all the prices charged by the other firm in all previous periods.

   That is, firm i matches the other firm's lowest price. Is the strategy pair (s_1, s_2) a NE of the infinitely repeated game for any discount factor less than 1?

   Solution: I figured you might want to see a nice textbook solution. This question is 454.3 from Osborne's textbook. And here's the solution:

a. Suppose that firm i uses the strategy s_i. If the other firm, j, uses s_j, then its discounted average payoff is

$$(1-\delta)\left(\tfrac{1}{2}\pi(p^m) + \tfrac{1}{2}\delta\pi(p^m) + \cdots\right) = \tfrac{1}{2}\pi(p^m).$$

If, on the other hand, firm j deviates to a price p then the closer this price is to p^m, the higher is j's profit, because the punishment does not depend on p. Thus by choosing p close enough to p^m the firm can obtain a profit as close as it wishes to $\pi(p^m)$ in the period of its deviation. Its profit during its punishment in the following k periods is zero. Once its punishment is complete, it can either revert to p^m or deviate once again. If it can profit from deviating initially then it can profit by deviating once its punishment is complete, so its maximal profit from deviating is

$$(1-\delta)\left(\pi(p^m) + \delta^{k+1}\pi(p^m) + \delta^{2k+2}\pi(p^m) + \cdots\right) = \frac{(1-\delta)\pi(p^m)}{1-\delta^{k+1}}.$$

Thus for (s_1, s_2) to be a Nash equilibrium we need

$$\frac{1-\delta}{1-\delta^{k+1}} \leq \tfrac{1}{2},$$

or

$$\delta^{k+1} - 2\delta + 1 \leq 0.$$

(This condition is the same as the one we found for a pair of k-period punishment strategies to be a Nash equilibrium in the *Prisoner's Dilemma* (Section 14.7.2).)

b. Suppose that firm i uses the strategy s_i. If the other firm does so then its discounted average payoff is $\frac{1}{2}\pi(p^m)$, as in part *a*. If the other firm deviates to some price p with $c < p < p^m$ in the first period, and maintains this price subsequently, then it obtains $\pi(p)$ in the first period and shares $\pi(p)$ in each subsequent period, so that its discounted average payoff is

$$(1-\delta)\left(\pi(p) + \tfrac{1}{2}\delta\pi(p) + \tfrac{1}{2}\delta^2\pi(p) + \cdots\right) = \tfrac{1}{2}(2-\delta)\pi(p).$$

If p is close to p^m then $\pi(p)$ is close to $\pi(p^m)$ (because π is continuous). In fact, for any $\delta < 1$ we have $2-\delta > 1$, so that we can find $p < p^m$ such that $(2-\delta)\pi(p) > \pi(p^m)$. Hence the strategy pair is not a Nash equilibrium of the infinitely repeated game for any value of δ.