1 Multivariate Calculus Review

Find the first partial derivative with respect to each argument.

(a) $f(x_1, x_2) = 5x_1 + 3x_2$

ANSWER I will use $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ to denote the partial derivative with respect to x_1 and x_2, respectively.

$$\frac{\partial f}{\partial x_1} = 5 \quad \text{and} \quad \frac{\partial f}{\partial x_2} = 3$$

(b) $f(x_1, x_2) = \sqrt{x_1 x_2}$

ANSWER

$$\frac{\partial f}{\partial x_1} = \frac{1}{2}\sqrt{\frac{x_2}{x_1}} \quad \text{and} \quad \frac{\partial f}{\partial x_2} = \frac{1}{2}\sqrt{\frac{x_1}{x_2}}$$

(c) $f(x_1, x_2) = \log x_1 + \log x_2$

ANSWER

$$\frac{\partial f}{\partial x_1} = \frac{1}{x_1} \quad \text{and} \quad \frac{\partial f}{\partial x_2} = \frac{1}{x_2}$$

2 The Marginal Rate of Substitution and Optimal Choices

2.1 Cobb-Douglas Functions

Peter's utility function for good x_1 and x_2 is given by $U(x_1, x_2) = (x_1)^4 \cdot x_2$. His current income is \$100 and the prices of x_1 and x_2 are \$20 and \$10, respectively.

(a) Express his marginal rate of substitution between x_1 and x_2.

ANSWER To figure out the MRS, we differentiate the utility function by x_1 and x_2 in order -

$$\text{MU}_1 = \frac{\partial U}{\partial x_1} = 4(x_1)^3 \cdot x_2$$
$$\text{MU}_2 = \frac{\partial U}{\partial x_2} = (x_1)^4$$

With these two marginal utilities, we can express the MRS by

$$\text{MRS} = -\frac{\text{MU}_1}{\text{MU}_2} = -\frac{4(x_1)^3 \cdot x_2}{(x_1)^4} = -\frac{4x_2}{x_1} \tag{2.1}$$

(b) Plot the indifference curve that passes through bundle (1,1). Find the MRS at this bundle, and depict it on the graph.

ANSWER Let me skip its graph but explain how to get its formula. Notice that by consuming the bundle (1,1) Peter enjoys utilities $U(1,1) = 1^4 \cdot 1 = 1$. By definition we know that every bundle on the indifference curve of our interest should deliver exactly 1 utility level. Pin down the utility level by 1 and solve the equation for x_2:

$$U(x_1, x_2) = (x_1)^4 \cdot x_2 \to 1 = (x_1)^4 \cdot x_2 \to x_2 = \frac{1}{(x_1)^4}$$

I believe you are able to draw the last function on the (x_1, x_2)-coordinates. To answer the next question, we plug in (1,1) for (x_1, x_2) in MRS equation (2.1) -

$$\text{MRS} = -\frac{4x_2}{x_1} \Rightarrow \text{MRS}_{(1,1)} = -\frac{4}{1} = -4,$$

which implies that the (instantaneous) slope of the indifference curve, which we figured out above, should be -4.

(c) Write down the two conditions for the optimal choice.

ANSWER The two secrets of Happiness require

$$(1) 20x_1 + 10x_2 = 100$$
$$(2) \frac{\text{MU}_1}{p_1} = \frac{\text{MU}_2}{p_2} \Rightarrow \frac{4(x_1)^3 \cdot x_2}{20} = \frac{(x_1)^4}{10}$$

Some tedious algebras reduce the second condition into $2x_2 = x_1$.

(d) Find the optimal bundle for him.

ANSWER In order to find out the optimal bundle, we plug in $2x_2$ for x_1 in the first condition(In fact, it is the budget constraint for Peter.) -

$$20(2x_2) + 10x_2 = 100 \Rightarrow 50x_2 = 100 \Rightarrow x_2 = 2 \tag{2.2}$$

Since we know $2x_2 = x_1$ from the second condition, we can easily get the demand for x_1 - $x_1 = 4$.

2.2 Perfect Complements

[2010 Spring Midterm 1] Michael always consumes three hamburgers x_1 along with one Coke x_2 (This is the only healthy combination of the two products!).

(a) Propose Michael's utility function that represents his preferences over hamburgers and Coke (function $U(x_1, x_2)$).

ANSWER $U(x_1, x_2) = \min\{x_1, 3x_2\}$

(b) In the commodity space (x_1, x_2), carefully depict his indifference curve and mark the optimal proportion line.

ANSWER Since the two goods constitute perfect complements for him, the corresponding indifference curve is of L-shape. The optimal proportion line follow from the utility function above - $x_2 = \frac{1}{3}x_1$. I skip the graph.

(c) Write down two secrets of happiness that determine his optimal choice. Explain economic intuition behind the conditions (one sentence for each secret).

ANSWER The associated secrets of happiness with perfect complements state

$$(1) p_1x_1 + p_2x_2 = m$$
$$(2) x_2 = \frac{1}{3}x_1.$$

The first condition involves the same intuition as above: since there is no options for tomorrow, there is no gain from saving. Furthermore, Michael likes both goods. Therefore, he should exhaust his income to maximize the utility. The second condition bears on the perfect complements - he always wants to consume the two goods in the proportion of 3 to 1.

(d) Find his optimal choice of x_1 and x_2 as a function of p_1, p_2, m. Is the choice interior for any price and income?

ANSWER From the second condition, we know $x_2 = \frac{1}{3}x_1$. Substitute $\frac{1}{3}x_1$ for x_2 in the budget constraint to get

$$p_1 x_1 + p_2 \frac{1}{3} x_1 = m.$$

And solve it for x_1 to get the demand for hamburgers - $x_1 = \frac{3m}{p_2+3p_1}$. Using the second condition again, we get the demand for Cokes - $x_2 = \frac{m}{p_2+3p_1}$. Since both x_1 and x_2 are strictly positive (as long as p_1, p_2, m are), it is interior.

2.3 Perfect Substitutes

Isobel considers peaches and nectarines to be perfect substitutes. She spends \$5 for these fruits every week. The price of peach is \$1 a pound and the price of nectarine is \$2 a pound.

(a) Graph two indifference curves - one passes through the bundle (1,2) and the other passes through (2,2).

ANSWER Since they are perfectly substitutable to her, the indifference curve is linear - a straight line with a negative slope. Bear it in mind that she likes both fruits so the indifference curve of (2,2) should be above that of (1,2).

(b) Provide with one utility function that represents her preference. What is MRS?

ANSWER Perfect substitutes imply she cares only about their total number, which leads to $U(x_1, x_2) = x_1 + x_2$.

(c) Find the optimal bundle, and mark it on the graph. Is it interior?

ANSWER Before writing off its secrets of happiness(we will deal with it next week), intuitively she will not buy nectarines. Basically, she cannot distinguish their difference - why she is willing to buy more expensive one? We denote by x_1 and x_2 the demand for peaches and for nectarines, respectively. In light of the intuition, the answer must be $x_1 = 5$ and $x_2 = 0$. Note that she can buy 5 peaches with \$5.

(d) Suppose that the price of peach increases to \$3 a pound. How it affects the optimal bundle?

ANSWER Now the peach is more expensive. With the same argument above, she consumes nectarines only. Hence $x_2 = 2.5$ but $x_1 = 0$.

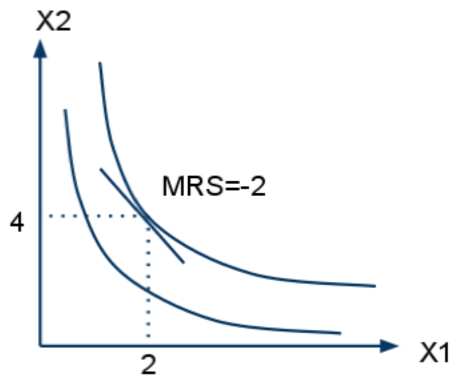
Econ 301 Final Exam Solutions - Spring 2011
Group A

Problem 1 - Consumer Choice

a) The budget constraint is $2x_1 + 4x_2 = 40$. The slope of the budget line is $-\frac{p_1}{p_2} = -\frac{2}{4} = -0.5$. Interpretation of the slope: the relative price of one tulip in the market is $\frac{1}{2}$ daffodil.

b) $U = \ln x_1 + \ln x_2$ or $U = x_1 x_2$ will represent the same preferences, since the operations of taking square root, adding a constant and taking the square of a function are all monotone transformations.

c)$MRS = -\frac{MU_{x_1}}{MU_{x_2}} = -\frac{x_2}{x_1}$. At $(2, 4)$ $MRS = -2$. Interpretation: given 2 tulips and 4 daffodils, Jeremy is willing to trade 1 tulip for 2 daffodils.



d) The two secrets of happiness are

$$2x_1 + 4x_2 = 40$$
$$-\frac{x_2}{x_1} = -0.5$$

The first condition implies the optimal bundle lies on the budget line. The second condition guarantees the indifference curve is tangent to the budget line at optimum. (MRS equals the slope of the budget line)
The optimal bundle is $(x_1, x_2) = (10, 5)$. The solution is interior since $x_1 \neq 0$ and $x_2 \neq 0$.

e) The function $U = 2x_1 + 3x_2$ represents preferences over perfect substitutes. $\frac{MU_1}{p_1} = \frac{2}{2} = 1 > \frac{MU_2}{p_2} = \frac{3}{4}$. Hence good 1 only will be consumed, and optimal $(x_1, x_2) = (20, 0)$. The solution is not interior.