ECON 455, Discussion Section 10

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Office: SS 6470. OH: Wed 8:00-9:30am; Thu 4:15-5:45pm; or by appt.

1. Find the mixed Nash equilibrium in the following game. *Note: We're doing this to to see how similar it is to finding an interim optimal mixed strategy for a forgetful driver, which we'll do next. Hint: To be willing to mix across actions, a player must be indifferent between those two actions – exploiting this allows us to solve for the mixed Nash equilibrium.*

	B	D
A	1,0	0,3
C	0,2	4,0

Solution:

You can first underline best responses and convince yourself that no pure-strategy Nash equilibrium exists. The game is actually just a modified version of matching pennies (which, as a game of pure conflict, has no pure-strategy NE). In terms of finding the mixed stategy NE, let's suppose P1 is putting weight α on A and $(1-\alpha)$ on C while P2 is putting weight β on B and $(1-\beta)$ on D. Then, using the hint, in the mixed NE P1 must be indifferent between playing A and C. This implies:

$$\begin{aligned} U_1(A, \beta B + (1-\beta)D) &= U_1(C, \beta B + (1-\beta)D) \\ \beta \times 1 + (1-\beta) \times 0 &= \beta \times 0 + (1-\beta) \times 4 \\ \beta &= 4(1-\beta) \\ \beta &= \frac{4}{5} \end{aligned}$$

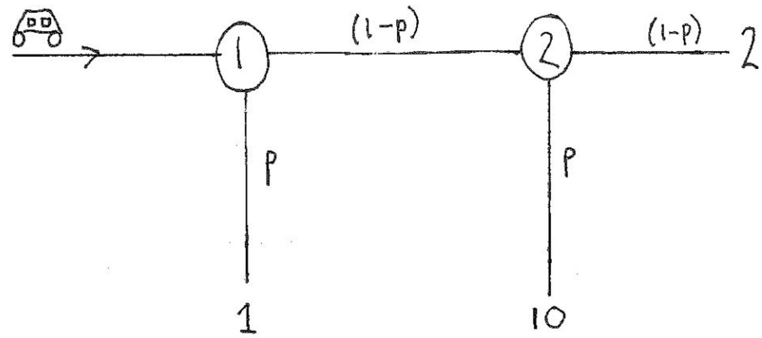
Note that in ensuring that P1 was indifferent between his two actions (which was necessary for him to be willing to put positive weight on each), we determined that P2 must, in the mixed NE, be putting weight $\frac{4}{5}$ on B and $\frac{1}{5}$ on D. This is the counter-intuitive part of solving for mixed NE. P1's indifference pins down P2's mix. And, of course, P2's indifference pins down P1's mix:

$$\begin{aligned} U_2(B, \alpha A + (1-\alpha)C) &= U_2(D, \alpha A + (1-\alpha)C) \\ \alpha \times 0 + (1-\alpha) \times 2 &= \alpha \times 3 + (1-\alpha) \times 0 \\ 3\alpha &= 2(1-\alpha) \\ \alpha &= \frac{2}{5} \end{aligned}$$

Therefore we can conclude that the mixed NE is

$$\left(\left(\frac{2}{5}, \frac{3}{5}\right), \left(\frac{4}{5}, \frac{1}{5}\right)\right)$$

2. Consider a forgetful driver (doesn't know which intersection he is at) whose payoffs are as described in the picture below:



(a) Find the *ex ante* optimal pure strategy (i.e. $p = 1$ or $p = 0$).

Solution: Remember that *ex ante* means that he's deciding this probability before he sets off on his drive. If $p = 1$, he will get a payoff of 1. If $p = 0$, he will get a payoff of 2, therefore $p = 0$ is the optimal pure strategy.

(b) Find the *ex ante* optimal mixed strategy.

Solution: Now he wants to maximize the following:

$$1p + 10p(1 - p) + 2(1 - p)^2$$

If we take the derivative and set it equal to zero we get $p^* = \frac{7}{16}$.

(c) Is $p = 1$ interim optimal? Show why or why not.

Solution: Remember that the interim concerns how we feel about our plan once we actually show up at an intersection (could be either the first or the second). If we know our strategy is that $p = 1$, then as we show up at an intersection we know it is the first intersection, because we would never make it past the first intersection. Given that, and given that we assume we're going to follow our plan in the future, we have an incentive to deviate and go straight rather than turn, because then we'll get 10 instead of 1 (because $p = 1$ implies we'll definitely turn right at the next intersection). Therefore $p = 1$ is not interim optimal.

(d) Is $p = 0$ interim optimal? Show why or why not.

Solution: If $p = 0$, then our path always takes us through both intersections, so when we show up at an intersection, we think it is equally likely that we're at either of the two intersections. Then, if we choose to deviate (by turning right), we'll get 1 with a probability of a half and 10 with a probability of a half, giving us an expected payoff of $11/2$. Since $11/2 > 2$, we do have an incentive to deviate from our plan and turn. Therefore $p = 0$ is not interim optimal.

(e) Find an interim optimal $p \in (0, 1)$. Compare it to the *ex ante* optimal mixed strategy.

Solution: To find an interim optimal $p \in (0, 1)$ is much the same as finding a mixed Nash equilibrium. Our strategy needs to be such that, given our strategy, we're indifferent between turning right and going straight. Note that if we're turning right with probability p, the the probability we're at the first and second intersection respectively is as follows:

$$\mu(h_1|X) = \frac{Pr^\sigma(h_1)}{Pr^\sigma(h_1) + Pr^\sigma(h_2)} = \frac{1}{1+(1-p)} = \frac{1}{2-p}$$
$$\mu(h_2|X) = \frac{Pr^\sigma(h_2)}{Pr^\sigma(h_1) + Pr^\sigma(h_2)} = \frac{1-p}{1+(1-p)} = \frac{1-p}{2-p}$$

This is identical to how it was in the lecture notes, if you remember. Now, what our his expected payoffs (EU) from the two actions. If he is truly willing to follow this mixed strategy (i.e. $p \in (0,1)$), then these must be equal, so let's set them equal:

$$EU(\text{straight}|p) = EU(\text{right}|p)$$
$$\mu(h_1|X)(10p + 2(1-p)) + \mu(h_2|X)2 = \mu(h_1|X)1 + \mu(h_2|X)10$$
$$\frac{1}{2-p}(10p + 2(1-p)) + \frac{1-p}{2-p}2 = \frac{1}{2-p}1 + \frac{1-p}{2-p}10$$
$$10p + 2(1-p) + 2(1-p) = 1 + 10(1-p)$$
$$p = \frac{7}{16}$$

So $p = \frac{7}{16}$ is interim optimal. Note that this is identical to the *ex ante* optimal mixed strategy, which should not surprise us as it was a theorem given in the lecture.

3. You are able to go to two stores to purchase a particular television you want to buy. Your prior belief is that, at each store, the TV could be priced anywhere between 100 and 300 dollars, and any dollar amount in that range is equally likely. Also, you believe that the pricing between the stores is independent, so even if the price at one store is 100, it is still just as likely that the price at the other store will be 300 (or any other $p \in (100, 300]$) as it will be 100. Unfortunately, you're a bit spacey and can't remember the exact price at the first store when you visit the second – you can only remember whether you perceived the price as low or high, and have to decide whether to go back and purchase at the first store or purchase at the second given that memory. You must purchase it from one of the two stores. Assume you're risk-neutral.

   (a) Suppose you define $p_1 \in [100, 250)$ as low and $p_1 \in [250, 300]$ as high. Further suppose that you're now at the second store and you remember that the price at the first store was high. For what p_2 will you choose to purchase from the second store?

   Solution: You'll purchase at the second store if the price is lower than the expected price at the first score. That is, you'll purchase at store 2 if:

$$p_2 \leq E[p_1|p_1 \text{ is high}]$$
$$p_2 \leq E[p_1|p_1 \in [250, 300]]$$
$$p_2 \leq 275$$

   (b) What if you instead remember that the price at the first store was low? For what p_2 will you choose to purchase from the second store?

   Solution: Same thing again:

$$p_2 \leq E[p_1|p_1 \text{ is low}]$$
$$p_2 \leq E[p_1|p_1 \in [100, 250)]$$
$$p_2 \leq 175$$

(c) What is your *ex ante* expected payment given your partitioning of the price space (i.e $p_1 \in [100, 250)$ is low and $p_1 \in [250, 300]$ is high)?

Solution: The probability that you perceive the first price as low is $P(p_1 < 250) = 3/4$. Obviously then the probability you perceive the first price as high is $1/4$. If you perceive the first price as high, then you would purchase at store two if $p_2 \leq 275$ which occurs with probability $P(p_2 \leq 275) = 7/8$. In this case, the expected payment is $E[p_2|p_2 \in [100, 275]] = 375/2$. And with probability $1/8$, $p_2 \geq 275$ and you go back to store 1 and pay an expected price of 275. That gives us the first part of the expected payment (the part coming from perceiving the first price as high)

$$\text{Expected Payment} = \frac{1}{4}\left(\frac{7}{8} \times \frac{375}{2} + \frac{1}{8} \times 275\right) + \ldots$$

We now need to add on what happens when we perceive p_1 as low, which happens with probability $3/4$. We would choose to purchase at store 2 if $p_2 \leq 175$ which happens with probability $3/8$, in which case the expected payment is $\frac{100+175}{2} = \frac{275}{2}$. With complementary probability $5/8$, we go back and purchase at store one where the expected payment is 175. Adding that in gives us the following:

$$\text{Expected Payment} = \frac{1}{4}\left(\frac{7}{8} \times \frac{375}{2} + \frac{1}{8} \times 275\right) + \frac{3}{4}\left(\frac{3}{8} \times \frac{275}{2} + \frac{5}{8} \times 175\right) = \frac{2725}{16}$$

(d) Recall from the lectures that to find the optimal 2-category memory, we solve

$$\lambda^* = \mathbf{E}[p_2|p_2 \in (E_L, E_H)]$$

where λ^* was the optimal cutoff price between low and high, and E_L and E_H were the average prices amongst those classified as low and high respectively. Solve for the optimal 2-category memory (i.e. a partitioning of $[100, 300]$ into two intervals).

Solution: If λ^* is the cutoff, then the low interval is $[100, \lambda^*]$ and the high interval is $[\lambda^*, 300]$. Therefore $E_L = \frac{100+\lambda^*}{2}$ and $E_H = \frac{\lambda^*+300}{2}$. Plugging these in gives us:

$$\lambda^* = \mathbf{E}\left[p_2|p_2 \in \left(\frac{100+\lambda^*}{2}, \frac{\lambda^*+300}{2}\right)\right]$$

Now, given that p_2 is drawn from a uniform distribution on $[100, 300]$, but we're conditioning on it being between $\frac{100+\lambda^*}{2}$ and $\frac{\lambda^*+300}{2}$, now think anywhere between those bounds is equally likely, so the expectation of p_2 given that condition is just the midpoint of those bounds, i.e.

$$\lambda^* = \mathbf{E}\left[p_2|p_2 \in \left(\frac{100+\lambda^*}{2}, \frac{\lambda^*+300}{2}\right)\right] = \frac{\frac{100+\lambda^*}{2} + \frac{\lambda^*+300}{2}}{2}$$

Now we simply solve for λ^*:

$$\begin{aligned}
\lambda^* &= \frac{\frac{100+\lambda^*}{2} + \frac{\lambda^*+300}{2}}{2} \\
4\lambda^* &= 100 + \lambda^* + \lambda^* + 300 \\
2\lambda^* &= 400 \\
\lambda^* &= 200
\end{aligned}$$

So the optimal 2-category memory is low $\equiv [100, 200]$ and high $\equiv (200, 300]$. This is actually basically the same problem we did in class except in class we were talking about $[0, 1]$ whereas here we're talking about $[100, 300]$, but you shouldn't be surprised that given we had uniform distributions in either case that the optimal 2-category memory involved splitting up the space exactly in the middle.

(e) What is your *ex ante* expected payment given the optimal 2-category memory you found in (d)? Compare it to that in (c).

Solution: Using identical analysis to that in part (c) (except with the cutoff at 200) gives us:

$$\text{Expected Payment} = \frac{1}{2}\left(\frac{3}{4} \times \frac{350}{2} + \frac{1}{4} \times 250\right) + \frac{1}{2}\left(\frac{1}{4} \times \frac{250}{2} + \frac{3}{4} \times 150\right) = \frac{675}{4}$$

$\frac{675}{4} = 168\frac{3}{4}$ and $\frac{2725}{16} = 170\frac{5}{16}$ as mixed fractions, and, as you can see, we *ex ante* expect to pay slightly less when we have our 2-category memory set optimally.