ECON 455, Discussion Section 11

TA: Shane Auerbach (sauerbach@wisc.edu) ; Date: 04/24/15

Office: SS 6470. OH: Wed 8:00-9:30am; Thu 4:15-5:45pm; or by appt.

1. (Transference) Suppose people lump all vehicles (in particular, airplanes (a), boats (b), and cars (c)) into one product category. Further suppose there are two types of consumers, which we'll refer to as the "unwashed masses (M)" and the "one-percenters (O)" respectively. While both groups lump together the three types of vehicles, the three vehicles have different salience levels to each group as follows:

$$p^M = (p_a^M, p_b^M, p_c^M) = \left(0, \frac{1}{4}, \frac{3}{4}\right), \quad p^O = (p_a^O, p_b^O, p_c^O) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

Let q_a, q_b, and q_c denote the quality of an airplane, boat and car respectively, and consider that they are a function of wing length (x), pontoon circumference (y) and radio antenna length (z) respectively in the following way:

$$q_a = 3x, \quad q_b = 2y, \quad q_c = z$$

Assume that each vehicle will sell for its perceived quality, and that the costs of building wings, pontoons and radio antennas are:

$$c(x) = x^2, \quad c(y) = 2y^2, \quad c(z) = 3z^2$$

(a) For just this part, assume that agents actually don't group the three vehicles. Find the optimal x, y, z (to maximize profits) for each of a, b, and c.

Solution: Note immediately that we're not going to be putting wings and pontoons on cars because they are costly and don't affect the quality (and therefore the price) of the cars. For the airplane, we want to build wings up until the point where the marginal cost of extending the wing equals the marginal benefit. That is, we set $3 = 2x \Rightarrow x^* = 3/2$. For the boat, we set $2 = 4y \Rightarrow y^* = 1/2$ and for the car we set $1 = 6z \Rightarrow z^* = 1/6$.

(b) Going forward, assume the agents do group the three vehicles into one category. Explain why we might want to put pontoons and/or wings onto cars?

Solution: If people group the three modes of transport together, then they associate bigger pontoons and longer wings with better boats and planes, and since they group the three, they transfer those properties onto cars also. By putting pontoons and wings on cars, we may increase the value of those cars due to this transference, even though the cars are actually not made better off by pontoons and wings.

(c) You are asked to design the optimal car (in that it maximizes profits) to sell to the masses. What is it?

Solution: The masses put salience $p_b^M = 1/4$ on boats and $p_c^M = 3/4$ on cars. Therefore, we want to choose x, y, z to solve:

$$\begin{aligned} &\max_{x,y,z} 0q_a(x) + \frac{1}{4}q_b(y) + \frac{3}{4}q_c(z) - c(x) - c(y) - c(z) \\ =&\max_{x,y,z} \frac{1}{4}2y + \frac{3}{4}z - x^2 - 2y^2 - 3z^2 \end{aligned}$$

Obviously this is strictly decreasing in x so $x^* = 0$ is optimal. As for y^* and z^*, we can simply take a first order conditions:

$$(FOC_y)\frac{1}{2} - 4y = 0 \Rightarrow y^* = \frac{1}{8}$$
$$(FOC_z)\frac{3}{4} - 6z = 0 \Rightarrow z^* = \frac{3}{24} = \frac{1}{8}$$

Therefore the profit-maximizing car we would design for the masses has a pontoon of circumference 1/8 (even though this does not improve the quality of the car) and a radio antenna length of 1/8.

(d) Find the optimal (in that it maximizes profits) boat to sell to the masses (hint: you do not need to do any math).

Solution: It's just the same as the optimal car. You can try to set it up and you'll see you're writing the same equation, since the buyers group the two products in any case.

(e) Find the optimal airplane/boat/car to sell to the one-percenters (O).

Solution: Now we're trying to solve the following:

$$\max_{x,y,z} \frac{1}{2}q_a(x) + \frac{1}{4}q_b(y) + \frac{1}{4}q_c(z) - c(x) - c(y) - c(z)$$
$$= \max_{x,y,z} \frac{1}{2}3x + \frac{1}{4}2y + \frac{1}{4}z - x^2 - 2y^2 - 3z^2$$

We can take three first order conditions to find:

$$(FOC_x)\frac{3}{2} - 2x = 0 \Rightarrow x^* = \frac{3}{4}$$
$$(FOC_y)\frac{1}{2} - 4y = 0 \Rightarrow y^* = \frac{1}{8}$$
$$(FOC_z)\frac{1}{4} - 6z = 0 \Rightarrow z^* = \frac{1}{24}$$

(f) Finally, suppose half of the population are type O (as illogical as that may be) and half are type M and you can only design one boat that you must sell to both types. Find the optimal (i.e. profit-maximizing) boat.

Solution: Now we're trying to solve the following:

$$\max_{x,y,z} \frac{1}{2}\left(\frac{1}{2}q_a(x) + \frac{1}{4}q_b(y) + \frac{1}{4}q_c(z)\right) + \frac{1}{2}\left(\frac{1}{4}q_b(y) + \frac{3}{4}q_c(z)\right) - c(x) - c(y) - c(z)$$
$$= \max_{x,y,z} \frac{1}{4}3x + \frac{1}{4}2y + \frac{1}{2}z - x^2 - 2y^2 - 3z^2$$

We can take three first order conditions to find:

$$(FOC_x)\frac{3}{4} - 2x = 0 \Rightarrow x^* = \frac{3}{8}$$
$$(FOC_y)\frac{1}{2} - 4y = 0 \Rightarrow y^* = \frac{1}{8}$$
$$(FOC_z)\frac{1}{2} - 6z = 0 \Rightarrow z^* = \frac{1}{12}$$

(g) Suppose that agents can actually tell the difference between boats and airplanes, but aren't sure whether a car is really closer to an airplane or a boat. A recent study came out that said that air travel was significantly safer and a lot more fun than boating, even more so than thought previously. If you were building cars, describe (qualitatively) how this might affect your choice of x, y, z? Use a framing argument.

Solution: Slap some wings on that car and call it a 747 to frame it as an airplane rather than a boat. Reduce the pontoon size to zero, to make sure it's not viewed as a boat (and save costs). It won't really affect the size of the radio antenna you choose.

2. Martha is a quasi-hyperbolic discounter with $\beta = 1/5$ and $\delta = 1$. Her true ability, θ may be either $\theta = 1$ or $\theta = 3$ and she can put effort $e \in [0, 1]$ into studying for a course. If she does well in the course, it will generate a future payoff of 5. If she does poorly, it will generate a future payoff of 0. The probability she does well is $\frac{\theta e}{4}$. The cost of expending effort $e \in [0, 1]$ is ce, where she discovers c during the course but she believes beforehand that c is drawn from a uniform distribution on $[0, 1]$.

(a) Let $\pi_a(\theta, e, c)$ denote her expected payoff at the time of enrollment. Find it:

Solution:

$$\pi_a(\theta, e, c) = \beta \left(\frac{\theta e}{4} 5 - ce \right)$$

(b) Argue that, given our assumptions, $\pi_a(\theta, e, c)$ is increasing in e so $e^* = 1$ is optimal.

Solution: Note that $\frac{5}{4}\theta e > ce$ given our assumptions, so the positive term is always increasing faster in e than the negative term is decreasing.

(c) Let $\pi_b(\theta, e, c)$ denote her expected payoff during the course. Find it:

Solution:

$$\pi_b(\theta, e, c) = -ce + \beta \left(\frac{\theta e}{4} 5 \right)$$

(d) Let $E[\theta] = \hat{\theta}$. Obviously $\hat{\theta} \in [1, 3]$, and where it lies exactly depends on how Martha believes it's distributed. Find the optimal e^* for every possible $\hat{\theta}$, and c.

Solution: You always want to set $e^* = 1$ if the positive term outweighs the negative, and $e^* = 0$ otherwise. That is, $e^* = 1$ if

$$\frac{1}{5} \frac{\hat{\theta}}{4} 5 \geq c$$
$$\hat{\theta} \geq 4c$$

Otherwise $e^* = 0$ is optimal.

(e) Suppose $E[\theta] = \hat{\theta} = 5/2$. What probability does enrollment-Martha put on course-Martha exerting effort $e = 1$?

Solution: She will exert $e = 1$ if it is profitable to do so, i.e. if $4c \leq 5/2 \Rightarrow c \leq 5/8$, which occurs with probability $5/8$ since c is uniformly distributed on $[0, 1]$.

(f) $E[\theta] = \hat{\theta} = 5/2$ implies that $P[\theta = 3] = \frac{3}{4}$ and $P[\theta = 1] = \frac{1}{4}]$. Suppose Martha could take a costless test at enrolment time which would reveal θ to her during the course (before she chooses e). With probability 3/4 it will reveal $\theta = 3$ and with probability 1/4 it will reveal $\theta = 1$. If Martha signs up for the test, what probability does enrollment-Martha put on course-Martha exerting effort $e = 1$? Should Martha take the test?

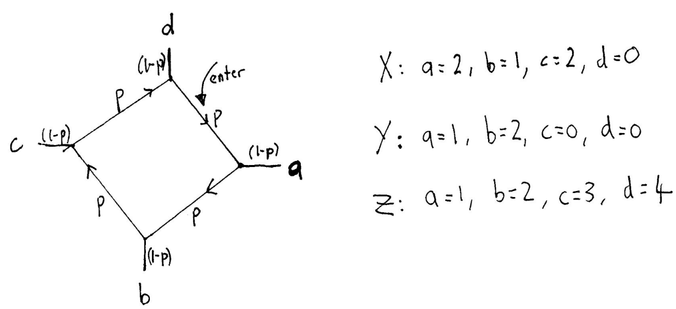
Solution: Well, the test result is $\theta = 3$ with probability 3/4, and in that case she sets $e = 1$ if $3 \geq 4c$, i.e. if $c \leq \frac{3}{4}$ which happens with probability 3/4. The test result could also be $\theta = 1$ with probability 1/4, and in that case she sets $e = 1$ if $1 \geq 4c$, i.e. if $c \leq \frac{1}{4}$. Therefore the total probability she sets $e = 1$ if she takes the test is:

$$P(e = 1) = \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{5}{8}$$

.

It's the same probability, so given that enrollment-Martha wants course-Martha to set $e = 1$, she is indifferent between signing up for the test and not.

3. (Forgetful driver challenge question) You're driving clockwise around a roundabout (traffic circle). You're a bit forgetful in that you can't tell the four exits on the roundabout from eachother. You know where you got on, but you can't see it each time you pass it, so you can't use that as a point of reference, and you don't remember how many exits you've already passed. Since you can't tell the exits apart, all you can do is choose p where, at each exit, p is the probability you stay on the roundabout and $(1 - p)$ is the probability you get off. All of this is shown in the picture below (albeit with a rather square roundabout). You're planning (in an *ex ante* sense) what p would be best for this situation.



Note that X, Y, Z describe three different iterations of this situation, each with different payoffs at exits a, b, c, d.

(a) Use intuition to find the optimal p for roundabout X.

Solution: He can get the maximal possible payoff by always exiting immediately, i.e. $p = 0$, so this is optimal. Note that

$$V(p) = (1 - p)2 + p(1 - p)1 + p^2(1 - p)2 + p^3(1 - p)0 + p^4 V(p).$$

Solve for V(p) on Wolfram Alpha and you'll get:

$$V(p) = \frac{2p^2 + p + 2}{p^3 + p^2 + p + 1}$$

If we then maximize $V(p)$ on $p \in [0, 1]$, again on Wolfram Alpha, we get $p^* = 0$.

(b) Use intuition to guess an approximately optimal p for roundabout Y.

Solution: Here if $p = 0$, then you're always getting 1, but you might be tempted to pick $p > 0$ so that you can sometimes get 2. We can actually argue that $p = 0$ is not *ex ante* optimal by noting that it isn't interim optimal, since the two have to coincide. If you were always playing $p = 0$, then given that you're at a turnoff, you know you're at the first turnoff (the one on the right). Then, by deviating to $p = 1$, assuming you'll stick with $p = 0$ in the future (the assumption we make when checking interim optimality, you could get a payoff of 2 for certain, and therefore there is an incentive to deviate and $p = 1$ is not interim optimal. If we want to actually solve it, we can compute it again. Note that

$$V(p) = (1-p)1 + p(1-p)2 + p^2(1-p)0 + p^3(1-p)0 + p^4V(p).$$

Solve for V(p) on Wolfram Alpha and you'll get:

$$V(p) = \frac{2p+1}{p^3+p^2+p+1}$$

If we then maximize $V(p)$ on $p \in [0,1]$, again on Wolfram Alpha, we get $p \approx 0.27$.

(c) Use intuition to argue that there is no optimal p for roundabout Z. But also argue that we know roughly what that optimal p should be.

Solution: The intuition here is a little more complicated. First, note that no matter what $p \in [0,1)$ we choose, we are more likely to to end up turning off at the intersections that come sooner than we are to get off at those that come later. So we're more likely to get 1 than we are 2, 2 more likely than 3 and 3 more likely than 4. The reasoning is simple: we get 1 with probability $(1-p)$. We get 2 with probability $p(1-p)$ and we know $p(1-p) < (1-p) \forall p \in (0,1)$. But note that as p gets closer to 1, the probability we get 2 actually approaches the probability we get 1. In fact, as p goes closer and closer to 1, we essentially expect to go round and round and round, so the degree to which 1 is more likely than 2 decreases and decreases. In the limit, as $p \to 1$, all four payoffs become equally likely, which is essentially what we would like, as the later (and therefore less likely for $p \in [0,1)$) payoffs are better. So the higher the p the better. However, if $p = 1$, are payoffs are actually undefined, because we never actually turn off. So, roughly speaking, the optimal p is a p that's as close to 1 as possible without equalling 1. Of course, that exact p does not exist. For any $p < 1, \exists p' s.t. p < p' < 1$, and that's the technical reason why no solution exists, even though we know roughly what it is.

If we want to do it using math, note that

$$V(p) = (1-p)1 + p(1-p)2 + p^2(1-p)3 + p^3(1-p)4 + p^4V(p).$$

We again turn to Wolfram Alpha to solve, giving us:

$$V(p) = \frac{4p^3+3p^2+2p+1}{p^3+p^2+p+1}$$

Then maximizing $V(p)$ with respect to p on $0 \le p \le 1$ through Wolfram Alpha gives us $p^* = 1$. Of course, if you force it to solve for $p < 1$, it actually maximizes numerically so suggests a p very close to 1 as being optimal. Given our discussion above, we know it's not technically optimal, even if it's very close!

(d) Set up the equation that you would maximize to solve for the optimal p.

Solution: I actually did this in the solution to each part above.