ECON 455, Discussion Section 5

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Office: SS 6470. OH: Wed 8:00-9:30am; Thu 4:15-5:45pm; or by appt.

1. (Cutoff strategies) A risk-neutral decision-maker (DM) has a delicious Twinkie, the eating of which will immediately generate 1 util. There is also a magical dessert genie who shows up every period with a dessert of quality q_t where each q_t is an i.i.d. random draw from $\left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$, with probability 1/3 on each possibility. In each period, the DM can eat either the dessert genie's offering (generating q_t utils), or the Twinkie (provided she has not already eaten it). The deliciousness continues infinitely into the future (and the Twinkie lasts forever, obviously), and the DM has discount rate δ.

Warning: You can theoretically do this all by hand, but it will destroy you. I recommend using WolframAlpha (WA) for the algebra, and I use it in the solutions.

(a) Let V^e denote the expected present value of entering a period having already eaten the Twinkie. Calculate V^e.

Solution: Once you've eaten the Twinkie, you're just going to eat whatever the genie brings you. Given that 1/4, 1/2, and 3/4 are equally likely, $\mathbf{E}[q_t] = 1/2$. But you get that in every period going forward, so

$$V^e = \frac{1}{2} + \frac{\delta}{2} + \frac{\delta^2}{2} + \ldots = \frac{1}{2}(1 + \delta + \delta^2) = \frac{1}{2(1-\delta)}.$$

(b) Let V_b^u denote the expected present value of entering a period with the Twinkie still uneaten/available given that the DM will eat the genie's offering if $q_t \in \{\frac{1}{2}, \frac{3}{4}\}$ but will eat the Twinkie instead if $q_t = \frac{1}{4}$. Calculate V_b^u.

Solution: Here there's a 1/3 chance you eat the Twinkie, and 2/3 you don't, i.e.

$$V_b^u = \frac{1}{3}(1 + \delta V^e) + \frac{1}{3}\left(\frac{1}{2} + \delta V_b^u\right) + \frac{1}{3}\left(\frac{3}{4} + \delta V_b^u\right)$$
$$V_b^u = \frac{1}{3}\left(1 + \frac{\delta}{2(1-\delta)} + \frac{1}{2} + \delta V_b^u + \frac{3}{4} + \delta V_b^u\right)$$
$$V_b^u = \frac{1}{3}\left(\frac{9}{4} + \frac{\delta}{2(1-\delta)} + 2\delta V_b^u\right)$$
$$V_b^u = \frac{9 - 7\delta}{4(\delta - 1)(2\delta - 3)}$$

You might have just avoided isolating V_b^u as we did in the last step if you thought the algebra was unpleasant. That's fine. When you eventually plug in a δ, you could do it in any version above and solve. If you do want to isolate V_b^u, I recommend outsourcing the algebra to WolframAlpha.com. Just plug in the following:

```
y=(1/3)(9/4+d/(2(1-d))+2d*y)
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Among many things it will do, it will solve for y, which here is our V_b^u.

(c) Let V_c^u denote the expected present value of entering a period with the Twinkie still uneaten/available given that the DM will eat the genie's offering if $q_t = \frac{3}{4}$ but will eat the Twinkie instead if $q_t \in \left\{\frac{1}{4}, \frac{1}{2}\right\}$. Calculate V_c^u.

Solution: Here there's a 2/3 chance you eat the Twinkie, and 1/3 you don't, i.e.

$$V_c^u = \frac{1}{3}(1+\delta V^e) + \frac{1}{3}(1+\delta V^e) + \frac{1}{3}\left(\frac{3}{4} + \delta V_c^u\right)$$
$$V_c^u = \frac{1}{3}\left(\frac{11}{4} + \frac{\delta}{1-\delta} + \delta V_c^u\right)$$
$$V_c^u = \frac{11-7\delta}{4(\delta-3)(\delta-1)}$$

(d) Let V_d^u denote the expected present value of entering a period with the Twinkie still uneaten/available given that the DM will eat the Twinkie no matter what. Calculate V_d^u.

Solution:

$$V_d^u = 1 + \delta V^e = 1 + \frac{\delta}{2(1-\delta)} = \frac{2-\delta}{2(1-\delta)}$$

(e) Supposing $\delta = 3/4$, what is the DM's optimal cutoff strategy?

Solution: Plugging $\delta = 3/4$ into the thing above should show you that V_c^u is the largest. Therefore the optimal cutoff strategy is to eat the Twinkie if and only if the dessert genie brings you one of the two lesser desserts.

(f) Find the optimal cutoff strategy for every $\delta \in (0,1)$. To do this, compare V_b^u, V_c^u and V_d^u to derive conditions on δ under which each is the largest of the three. Try working with WolframAlpha to see if you can get it to do the work for you.

Solution: You can plug in the following to find when V_b^u is the largest (you'll notice I'm replacing δ with x):

```
Simplify[(9-7x)/(4(x-1)(2x-3))>(2-x)/(2(1-x))&&(9-7x)/(4(x-1)(2x-3))>(11-7x)/(4(x-3)(x-1))&&0<x<1]
```

Doing operations like that for each of the three reveals that:

$$\max\{V_b^u, V_c^u, V_d^u\} = \begin{cases} V_b^u & \text{if } \delta \in \left[\frac{6}{7}, 1\right] \\ V_c^u & \text{if } \delta \in \left[\frac{1}{2}, \frac{6}{7}\right] \\ V_d^u & \text{if } \delta \in \left[0, \frac{1}{2}\right] \end{cases}$$

(g) Explain the intuition behind your answer to (f).

Solution: Remember that (b) said you ate the Twinkie only when the dessert genie brought you the worst possible dessert, (c) said you ate the Twinkie when the dessert genie brought you either of the two lesser desserts, and (d) said you always ate the Twinkie. Our answer in (f) shows that the as you get more patient, it becomes better to save the Twinkie for future consumption and accept the dessert genie's offering, as you might expect. If you're sufficiently impatient, i.e. low δ, then you should eat it immediately (i.e. the strategy in (d)), as we showed above.

(h) Now suppose that the dessert genie, in each period, brings the DM a dessert randomly drawn from the continuous uniform distribution over $[0,1]$. Let ϕ denote the DM's cutoff – for $q_t \leq \phi$, he eats the Twinkie while for $q_t > \phi$, he saves the Twinkie. Conjecture as to what you expect the optimal cutoff ϕ to be when $\delta = 0$ and also when $\delta = 1$, and explain your reasoning.

Solution: If we are totally impatient and don't value the future at all, then there's no point saving the Twinkie for the future. Therefore, we might as well pick a very high cutoff such that we definitely eat the Twinkie today. Therefore, the optimal cutoff should logically be $\phi^*(\delta = 0) = 1$. On the other hand, if you're infinitely patient, so that you care equally about all future periods as well as today, then you know that for any $q_t > 0$ that you might have in this period, it is inevitable that in a future period, call it t', you'll get $q_{t'} < q_t$, and you'll therefore prefer to save the Twinkie for that period rather than eat it today. Since this is true for any $q_t > 0$, the optimal cutoff should be $\phi^*(\delta = 1) = 0$.

(i) (Pretty hard!) Solve for the optimal cutoff, $\phi^*(\delta)$, i.e. that which maximizes V^u_ϕ for $\delta \in [0, 1]$. Compare your solution to the conjecture – graphing the solution is one nice way to compare.

Solution: You should be able to convince yourself that V^e hasn't really changed here because $\mathbf{E}[q_t]$ still equals 1/2.

$$\begin{aligned} V^u_\phi &= P(q_t < \phi)(1 + \delta V^e) + P(q_t > \phi)(\mathbf{E}[q_t | q_t > \phi] + \delta V^u_\phi) \\ &= \phi \left(1 + \frac{\delta}{2(1-\delta)}\right) + (1-\phi)\left(\frac{1+\phi}{2} + \delta V^u_\phi\right) \end{aligned}$$

Ask WolframAlpha:

```
solve for y, y = p(1+d/(2(1-d))) + (1-p)((1+p)/(2)+d*y)
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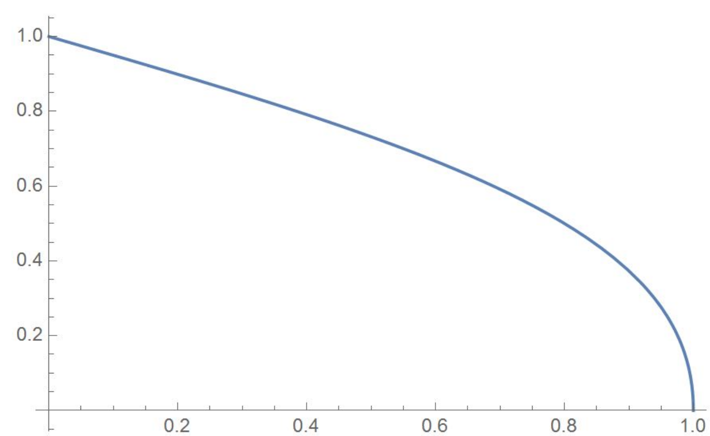
$$V^u_\phi = \frac{\delta\left(-\phi^2 + \phi + 1\right) + \phi^2 - 2p - 1}{2(\delta - 1)(\delta(\phi - 1) + 1)}$$

Finally, we want to choose ϕ to maximize this. Ask WolframAlpha!

```
d/dp (-1-2 p+p^2+d (1+p-p^2))/(2 (-1+d) (1+d (-1+p))) = 0
```

It looks like what I put in to WA is different, but it's equivalent. It's what WA gave me when I clicked on the previous result! Looking at WolframAlpha's solutions, we find that $\phi^*(\delta) = \frac{\sqrt{1-\delta^2}+\delta-1}{\delta}$. If we plot that, it looks like:

```
Plot[(Sqrt[1 - d^2] + d - 1) / d, {d, 0, 1}]
```



As you can see, it confirms the conjectures made in (h).

2. (Deal or no deal, modified from Angner 6.13 and 6.41) You are on the show *Deal or No Deal*, where you are facing some boxes, each of which contains an unknown amount of money. At this stage, you are facing three boxes. One of them contains \$900,000, one contains \$300,000, and one contains \$60, but you do not know which is which. If you choose to open the boxes, you can open them in any order you like, but you can keep the amount contained in the last box only. Assume you have no outside, pre-existing wealth.

Warning: This question is very easy. Don't expect exams to be this easy.

(a) What is the expected value of opening the three boxes?

Solution:

$$EV = \frac{900000 + 300000 + 60}{3} = 400020$$

(b) The host gives you the choice between a sure \$400,000 and the right to open the three boxes. Assuming you want to maximize expected value, which should you choose?

Solution: $400,020 > 400,000$ by my math, so you would reject the offer.

(c) You decline the \$400,000 and open a box. Unfortunately, it contains \$900,000. What is the expected value of opening the remaining two boxes?

Solution:

$$EV = \frac{300000 + 60}{2} = 150030$$

(d) The host gives you the choice between a sure \$155,000 and the right to open the remaining two boxes. Assuming you want to maximize expected value, what should you choose?

Solution: $150030 < 155000$, so you would accept the offer.

(e) For the remaining parts, assume there are actually three boxes with \$1,000,000, \$1,000 and \$10 respectively and the host is offering you \$250,000 to stop playing. Assuming that you want to maximize expected value, would you accept the offer?

Solution: The expected value is then the mean of those, i.e. 333670. Obviously that's greater than 250000, so you would not accept.

(f) Assuming your utility function is $u(x) = \sqrt{x}$, and that you want to maximize expected utility, would you accept the offer?

Solution: You will accept if

$$\begin{aligned} u(250000) &> \frac{1}{3}(u(1000000) + u(1000) + u(10)) \\ \sqrt{250000} &> \frac{1}{3}(\sqrt{1000000} + \sqrt{1000} + \sqrt{10}) \\ 500 &> \text{about } 345 \ \checkmark \end{aligned}$$

So yeah, you would accept.

(g) Given the utility function above, what is the lowest amount would you accept to forego your right to continue opening the boxes?

Solution: The lowest amount, x, that you would accept would be that which makes you indifferent. That is,

$$\sqrt{x} = \frac{1}{3}(\sqrt{1000000} + \sqrt{1000} + \sqrt{10})$$
$$x \approx 118976$$

3. (Angner 7.7) A person's value function is $v(x) = \sqrt{\frac{x}{2}}$ for gains and $v(x) = -2\sqrt{|x|}$ for losses. The person is facing the choice between a sure \$2 and a 50-50 gamble that pays \$4 if she wins and \$0 if she loses.

   (a) Show algebraically that this person is loss averse, in the sense that she suffers more when she loses \$4 than she benefits when she receives \$4.

   Solution: $v(-4) = -4$ while $v(+4) \approx 1.41$. Therefore, losing 4 costs her 4 utils while gaining 4 gives her only 1.41 utils.

   (b) If the outcomes are coded as gains, meaning that she will take the worst possible outcome as her reference point, what is the value of

   i. the sure amount?

   Solution: $v(+2) = 1$

   ii. the gamble?

   Solution: $v(G) = \frac{1}{2}v(0) + \frac{1}{2}v(+4) = \frac{1}{2}\sqrt{2} \approx 0.71$

   iii. Which would she prefer?

   Solution: The sure amount.

   (c) If the outcomes are coded as losses, meaning that she will take the best possible outcome as her reference point, what is the value of

   i. the sure amount?

   Solution: $v(-2) = -2\sqrt{2} \approx -2.83$

   ii. the gamble?

   Solution: $\frac{1}{2}v(0) + \frac{1}{2}v(-4) = -2$

   iii. Which would she prefer?

   Solution: The gamble.