ECON 455, Discussion Section 8

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Office: SS 6470. OH: Wed 8:00-9:30am; Thu 4:15-5:45pm; or by appt.

1. (Watson 7.1.c - Iterated deletion using mixes) Use iterated deletion of strictly dominated strategies to find the unique Nash equilibrium in the following game.

1\2	L	C	R
U	6,3	5,1	0,2
M	0,1	4,6	6,0
D	2,1	3,5	2,8

Solution:

You should be able to find pretty quickly that no pure action is strictly dominated by another mixed action. L, C and R are each unique best responses to U, M and D respectively, so you can further conclude that none of these is strictly dominated, even by a mix (think about why!). For P1, U is the unique best response to L and C and M is the unique best response to R, so they cannot be strictly dominated. This leaves just D, which intuitively might seem like it does *OK* against everything, but never particularly well. Let's see if a mix of U and M can strictly dominate it. Let $\alpha(U)$ be the weight on U, so the weight on M is, correspondingly $(1 - \alpha(U))$. Then, for D to be strictly dominated, it the mix of U and M beats D for each case, L, C and R, i.e.

$$\begin{aligned} \text{For } L: \alpha(U) \cdot 6 + (1-\alpha(U))0 > 2 \;&\Leftrightarrow\; \alpha(U) > 1/3 \\ \text{For } C: \alpha(U) \cdot 5 + (1-\alpha(U))4 > 3 \;&\Leftrightarrow\; \text{any } \alpha(U) \text{ works} \\ \text{For } R: \alpha(U) \cdot 0 + (1-\alpha(U))6 > 2 \;&\Leftrightarrow\; \alpha(U) < 2/3 \end{aligned}$$

Therefore, you can conclude that any mix of U and M that puts weight between 1/3 and 2/3 on U strictly dominates D, and therefore we can remove it from consideration. As an aside, in doing the above, we have found all possible mixes by which we can strictly dominated D, but this was perhaps a bit overkill for iterated deletion of strictly dominated strategies – we just need to find one mix that dominates it. Once you get a sense of these, you can probably just guess a mix that you think works and then check. For example, if you guess $\alpha(U) = 1/2$, then you can just check that the average of of U and M beats D for each P2 action.

With D eliminated from consideration, we're left with the following:

1\2	L	C	R
U	6,3	5,1	0,2
M	0,1	4,6	6,0

Now, R is strictly dominated by L, leaving us with:

1\2	L	C
U	6,3	5,1
M	0,1	4,6

Finally, M is strictly dominated by U and, subsequently, C is strictly dominated by L. Therefore, the unique Nash equilibrium is (U, L).

2. (Behavioral Pirates) Five pirates (Antonio, Beatriz, Carla, David and Elvin) have obained 100 gold doblónes and have to divide up the loot. Antonio proposes a distribution of the loot. All pirates vote on the proposal. If half or more agree, the loot is divided as proposed by Antonio. If Antonio fails to obtain support of at least half the the crew (including himself), then he will be killed, in which case the pirates start over with Beatriz as the proposer. If she gets half the remaining crew (including herself) to agree, then the loot is divided as she proposes. If not, then she is killed, and Carla then makes the proposal. If Carla's proposal is not agreed on by half the remaining crew, including herself, then she is killed. Finally, David makes a proposal – if it isn't accepted by half the crew, including himself, he is killed. If everybody but Elvin is killed, he takes all the loot. The pirates are infinitely sophisticated and rational (as all pirates are). When a pirate is indifferent between voting to kill or supporting a proposal, he votes to kill. You cannot split a doblóne, so proposals must be whole numbers of doblónes. For each of the following questions, solve using backward induction.

   (a) First, assume each pirate cares only about how many doblónes she ends up with (the more the better). What is the maximum number of doblónes Antonio can keep without being killed?

      Solution: When David proposes, he will propose that he keep all of the loot. Since he is automatically half of the remaining crew, he will receive all of the loot.

      Given this, when Carla proposes, she knows that Elvin is getting zero if he votes to kill her. Therefore, is she offers Elvin 1 doblóne and keeps the rest for herself, Elvin will vote for her and she will 99 doblónes.

      Given this, when Beatriz proposes, she knows that if she is killed, David will get nothing (because of Carla's proposal). Therefore, if Beatriz offers 1 doblóne to David, he will vote for her proposal. Therefore, she will offer 1 doblóne to David and keep 99 for herself.

      Given this, when Antonio proposes, he knows that if he is killed, Carla and Elvin will get nothing. Therefore, he can offer each of them a doblóne and they will vote for him. Therefore, Antonio can actually keep a maximum of 98 doblónes for himself!

   (b) Now, in a more *behavioral* approach, lets assume pirates quite like to see their colleagues killed. In particular, getting to see a colleague killed yields each the utility equivalent of x doblónes.

      i. What is the maximum Antonio can keep if $x = 2$?

         Solution:

         When David proposes, he will propose that he keep all of the loot. Since he is automatically half of the remaining crew, he will receive all of the loot.

         Given this, when Carla proposes, she knows that Elvin is getting just two (additional) utils if he votes to kill her. Therefore, if she offers Elvin 3 doblónes and keeps the rest for herself, Elvin will vote for her and she will get 97 doblónes.

         Given this, when Beatriz proposes, she knows that if she is killed, David will get just two (additional) utils (because of Carla's proposal and Beatriz's death). Therefore, if Beatriz offers 3 doblónes to David, he will vote for her proposal. Therefore, she will offer 3 doblónes to David and keep 97 for herself.

Given this, when Antonio proposes, he knows that if he is killed, Carla and Elvin will get 2 doblónes each (in utils). Therefore, he can offer each of them 3 doblónes and they will vote for him. Therefore, Antonio can actually keep a maximum of 94 doblónes for himself.

ii. What is the minimum x such that Antonio will definitely be killed, regardless of the offer he makes?

Solution: If we look at this generally, in each case Antonio just has to buy off two people to save his life. The cheapest two to buy off are the two that are going to get screwed by Carla. Since those two each value his death at x, he needs to offer each of them at least $x+1$ to preserve his own life. He can only save his life if $2x+2 \leq 100$, i.e. if $x \leq 49$. Therefore, $x = 50$ is the minimum x such that he dies regardless of his offer.

3. (Lemons (Akerlof 1970)) Suppose sellers in a used car market know whether their cars are good or bad but buyers do not and the valuations for buyers and sellers for the two types of cars are as follows:

	Good	Bad
Buyer	60	20
Seller	50	10

Suppose that a fraction α of the cars on the market are bad, and this is public information. Assume sellers accept offers if indifferent.

(a) Describe what it would mean for an outcome to be Pareto efficient in this game.

Solution: All it entails here is that the buyers end up with the cars, since they value them more. The prices the transactions take place at are irrelevant in terms of Pareto efficiency.

(b) If $\alpha = 1/6$, what should buyers offer for a car?

Solution: If they offer \$50, then all sellers will accept and they their expected utility is $\frac{5}{6}60 + \frac{1}{6}20 - 50 = \frac{10}{3}$. There's no point offering between \$10 and \$50 because seller who would accept $p \in [10, 50)$ would also accept $p = 10$. If you offer $p = 10$, then the seller of the bad car will agree, so the expected utility is $\frac{5}{6}0 + \frac{1}{6}(20-10) = \frac{5}{3} < \frac{10}{3}$. Any bid less than \$10 will get no interest, so utility of zero. Therefore the optimal bid is \$50.

(c) If $\alpha = 1/2$, what should buyers offer for a car?

Solution: Now if he offers \$50, his expected utility is $\frac{1}{2}60 + \frac{1}{2}20 - 50 = -10$, so that's not a great idea. If you bid 10 you get $\frac{1}{2}0 + \frac{1}{2}(20-10) = 5 > 0$. Therefore buyers should offer \$10. The market unravels here as the expensive cars do not change hands.

(d) How does this relate to the auction example we saw in class?

Solution: In the auction example in class, a potential buyer that valued the good (oil rights) higher than a seller could bid on an uncertain prospect. The seller knows the value of the prospect though. We found there that the potential buyer shouldn't even make an offer, because they will only win if the value of the prospect turns out to be such that they wouldn't want to win. This is the so-called *winner's curse*. The same thing happens here regarding bids between \$10 and \$50 – the transaction will only take place if the value of the car is such that the potential buyer actually doesn't want the transaction to take place.

4. (Estimating preferences) Consider the following Prisoner's Dilemma with the monetary payoffs, v_1 and v_2, listed, and suppose we know that i) people always believe their opponent will choose C and ii) always choose C themselves when $x > 6$ but always choose D when $x < 6$.

1\2	c	d
C	x,x	0,x+1
D	x+1,0	1,1

(a) Suppose $u_i(v_i, v_j) = v_i + \gamma v_j$, where $\gamma > 0$ represents how much people care about others. Calculate γ.

Solution:

In this case at $x = 6$, the person must be between choosing C and D given that the opponent is playing C. Therefore.

$$\begin{aligned} u_i(C, C|x = 6) &= u_i(D, C|x = 6) \\ 6 + 6\gamma &= 7 \\ \gamma &= 1/6 \end{aligned}$$

(b) Suppose $u_i(v_i, v_j) = v_i - \psi|v_i - v_j|$, where $\psi > 0$ represents how much people are bothered by *unfair* (unequal, really) outcomes. Calculate ψ.

Solution:

In this case at $x = 6$, the person must be between choosing C and D given that the opponent is playing C. Therefore.

$$\begin{aligned} u_i(C, C|x = 6) &= u_i(D, C|x = 6) \\ 6 - 0\psi &= 7 - 7\psi \\ \psi &= 1/7 \end{aligned}$$

(c) Is it possible that people hate one another and actually have $u_i(v_i, v_j) = v_i - kv_j$, with $k > 0$, given the information we have above? Why or why not?

Solution:

No, it is not possible because if they don't value their opponents pay-offs positively (or if they value it negatively), then C is a dominant strategy for them for $x = 6$. If we try to calculate a k such that it does work, we'll find that it's negative and is equivalent to the γ we found in (a).

$$\begin{aligned} u_i(C, C|x = 6) &= u_i(D, C|x = 6) \\ 6 - 6k &= 7 \\ k &= -1/6 \end{aligned}$$

5. (Rotten kid theorem (Becker 1974)) A child chooses an action that affects both his own income and his parent's income. He chooses $A > 0$ and incomes for child and parent are:

$$I_C(A) = A - A^2 \qquad\qquad I_P(A) = A - \frac{1}{2}A^2$$

What level of action A maximizes the child's income? Call this A_C^*. What level of action maximizes the joint income (i.e. the sum of the two incomes above)? Call this A_J^*.

Now suppose that the child does not care about his parent - he only cares about how much money he has. On the other hand, the parent does care about the child, and may choose to transfer an amount of money B to the child - think of this as a bequest. Their utility functions are as follows:

$$u_C = \sqrt{I_C(A) + B} \qquad\qquad u_P = \sqrt{I_P(A) - B} + \frac{1}{2}u_C$$

The child chooses A before the parent chooses B, and the parent observes A before choosing B. Use backward induction to solve this game. That is, you should first find how the parent best responds to the child's choice of A. Then, given this, you should find what A the child chooses. Once you have found the subgame perfect Nash equilibrium (SPNE) through backward induction, comment on how it compares to A_C^* and A_J^*.

Finally, try the problem again except consider the case in which the parent cares very little about his son. That is, replace u_P of above with $u'_P = \sqrt{I_P(A) - B} + \frac{1}{10}u_C$, holding everything else constant. How does this change the child's choice of A in the SPNE?

(Hint: This problem has a fair amount of algebra. And you'll need to use the chain rule in taking derivatives. Life is hard – get used to it.)

Solution:

First we must find A_C^*. Just take the derivative of $I_C(A)$, set it equal to zero, and solve to find $A_C^* = \frac{1}{2}$. Doing the same for $I_C(A) + I_P(A)$ gives $A_J^* = \frac{2}{3}$.

Now for the backwards induction. The parent will choose B to best respond to whatever A the child has chosen. They solve the following:

$$\max_B \sqrt{I_P(A) - B} + \frac{1}{2}\sqrt{I_C(A) + B}$$

The FOC is then

$$\begin{aligned}
-\frac{1}{2\sqrt{I_P(A) - B^*}} + \frac{1}{4\sqrt{I_C(A) + B^*}} &= 0 \\
2\sqrt{I_P(A) - B^*} &= 4\sqrt{I_C(A) + B^*} \\
I_P(A) - B^* &= 4(I_C(A) + B^*) \\
B^* &= \frac{I_P(A) - 4I_C(A)}{5}
\end{aligned}$$

Given that, the parent is playing B^*, the child solves

$$\begin{aligned}
&\max_A \sqrt{I_C(A) + B^*} \\
&= \max_A \sqrt{A - A^2 + \frac{I_P(A) - 4I_C(A)}{5}} \\
&= \max_A \sqrt{A - A^2 + \frac{A - \frac{1}{2}A^2 - 4(A - A^2)}{5}} \\
&= \max_A \sqrt{A - A^2 - \frac{3}{5}A + \frac{7}{10}A^2} \\
&= \max_A \sqrt{\frac{2}{5}A - \frac{3}{10}A^2}
\end{aligned}$$

We take a first order condition (using the chain rule):

$$\frac{1}{2\sqrt{\frac{2}{5}A^* - \frac{3}{10}A^{*2}}}\left(\frac{2}{5} - \frac{6}{10}A^*\right) = 0$$

$$\frac{2}{5} - \frac{6}{10}A^* = 0$$

$$A^* = \frac{2}{3}$$

This is the neat counter-intuitive result. Even though the child doesn't care about the income of the parent directly, he actually chooses the action that maximizes the joint income of parent and child, rather than just maximizing his own income.

Even more counter-intuitive is that the same thing happens even if the parent cares a lot less about the child (and therefore bequests a lot less). You can repeat the exercise with $u'_P = \sqrt{I_P(A) - B} + \frac{1}{10}u_C$. Do all the same steps and be careful with algebra – and you'll still find that $A^* = \frac{2}{3}$. In fact, this same result holds for any $u''_P = \sqrt{I_P(A) - B} + ku_C$, so long as $k > 0$. So as long as the parent cares about the child to some extent (which may be arbitrarily small), the SPNE results in the child choosing the action that maximizes joint income. Quite surprising, right? This is the *rotten child problem*, first analyzed by Becker (1974). He proved this result with more general conditions.