NU Econ 101 Lecture 11: Solving exchange economies

Lecturer: Shane Auerbach

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Where are we now?

Confessing to a lie

I introduced four types of utility function:

- Perfect substitutes $u(x, y) = x + y$.
- Perfect complements $u(x, y) = \min(x, y)$.
- Cobb-Douglas $u(x, y) = x^{\frac{1}{2}} \cdot y^{\frac{1}{2}}$.
- Logarithmic $u(x, y) = \ln(x) + \ln(y)$.

But, actually, Cobb-Douglas and logarithmic are equivalent!

- We usually call both Cobb-Douglas.

Equivalence of utility functions (1)

We will say two utility functions are **ordinally equivalent** if the two utility functions always lead to the same choices.

Formally, u_1 and u_2 are ordinally equivalent if

$$u_2 = f(u_1),$$

where f is any monotonic increasing transformation.

- Seems technical, but it really isn't.

Equivalence of utility functions (2)

Examples:

1. Suppose $u_1(x, y) = x + y$. For our monotonic increasing transformation, let's just double it:

$$u_2(x, y) = f(u_1(x, y)) = 2 \cdot u_1(x, y) = 2x + 2y$$

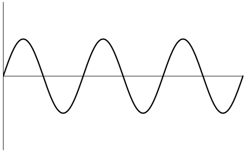
Equivalent: $u_1(x, y) = x + y$ and $u_2(x, y) = 2x + 2y$

Equivalence of utility functions (3)

2. Suppose $u_1(x, y) = x + y$. For our monotonic increasing transformation, let's just take $f = \sin()$:

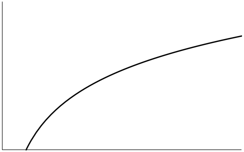
$$u_2(x, y) = f(u_1(x, y)) = \sin(u_1(x, y)) = \sin(x + y)$$

NO. BAD! sin() is not monotonic increasing!



Equivalence of utility functions (4)

3. Suppose $u_1(x, y) = \sqrt{x}\sqrt{y} = x^{\frac{1}{2}} \cdot y^{\frac{1}{2}}$, (C-D!). For our monotonic increasing transformation, let's just take $f = \ln()$. Is this allowed?



Yes! It's monotone increasing!

$$u_2(x, y) = f(u_1(x, y)) = \ln(u_1(x, y)) = \ln\left((x \cdot y)^{\frac{1}{2}}\right)$$

OK – still doesn't look like logarithmic utility ...

Equivalence of utility functions (5)

3. ctd. So far we have $u_2(x, y) = \ln\left((x \cdot y)^{\frac{1}{2}}\right)$.

$$
\begin{aligned}
u_2(x, y) &= \ln\left((x \cdot y)^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\ln((x \cdot y)) \\
&= \frac{1}{2}(\ln(x) + \ln(y)) \\
&= \frac{1}{2}\ln(x) + \frac{1}{2}\ln(y)
\end{aligned}
$$

Great – we have that $u_1(x, y) = \sqrt{x}\sqrt{y}$ is ordinally equivalent to $u_2(x, y) = \frac{1}{2}\ln(x) + \frac{1}{2}\ln(y)$. And ...

Equivalence of utility functions (6)

3. ctd. Remember that doubling the utility function is also a monotone transformation. So these three are all ordinally equivalent with eachother:
   - $u_1(x, y) = \sqrt{x}\sqrt{y}$
   - $u_2(x, y) = \frac{1}{2}\ln(x) + \frac{1}{2}\ln(y)$
   - $u_3(x, y) = \ln(x) + \ln(y)$

Equivalence of utility functions (7)

Let's do a new, more complicated, example:

4. Suppose $u_1 = x^{\frac{1}{3}} \cdot y^{\frac{2}{3}}$. Let's take the ln() for an ordinally equivalent one:

$$\begin{aligned} u_2(x, y) &= \ln\left(x^{\frac{1}{3}} \cdot y^{\frac{2}{3}}\right) \\ &= \ln\left(x^{\frac{1}{3}}\right) + \ln\left(y^{\frac{2}{3}}\right) \\ &= \frac{1}{3}\ln(x) + \frac{2}{3}\ln(y) \end{aligned}$$

And that is just ordinally equivalent (tripling it) to $u_3(x, y) = \ln(x) + 2\ln(y)$.

Equivalence of utility functions (8)

- $u_1(x, y) = x^{\frac{1}{3}} \cdot y^{\frac{2}{3}}$
- $u_3(x, y) = \ln(x) + 2\ln(y)$

But don't take my word for it. If u_1 and u_3 are ordinally equivalent, you should make the **same decisions** regardless of which you use.

Remember the Golden Rule for consumption for Cobb-Douglas and Logarithmic is to solve two equations:

1. The BC.
2. Equate slope of IC and BC (or equate bang-per-buck).

BC is independent of preferences, so **compare second equations**.

Equivalence of utility functions (9)

Comparing second equations:

$$u_1(x,y) = x^{\frac{1}{3}} \cdot y^{\frac{2}{3}}$$

$$\frac{MU_x(x,y)}{p_x} = \frac{MU_y(x,y)}{p_y}$$

$$\frac{y^{\frac{2}{3}}}{3x^{\frac{2}{3}} p_x} = \frac{2x^{\frac{1}{3}}}{3y^{\frac{1}{3}} p_y}$$

$$y \cdot p_y = 2x \cdot p_x$$

$$u_3(x,y) = \ln(x) + 2\ln(y)$$

$$\frac{MU_x(x,y)}{p_x} = \frac{MU_y(x,y)}{p_y}$$

$$\frac{1}{xp_x} = \frac{2}{yp_y}$$

$$y \cdot p_y = 2x \cdot p_x$$

Same thing!

Why do we care?

One less type of utility function!

Easier to differentiate the log format than with powers.

- If I give you one with powers, turn it into a log and then differentiate! Or not. Whatevs . . .

We get to have our cake and eat it too, or whatever.

- Sorry for pretending they were different. It allowed me to give you essentially the same problem twice without you noticing!

Where are we now?

Edgeworth boxes (1)

This is going to blow your mind!

- What if we plotted two agents' indifference curves **on the same plot!** And let's flip one upside-down!
  - We'll call this madness an **Edgeworth box**.

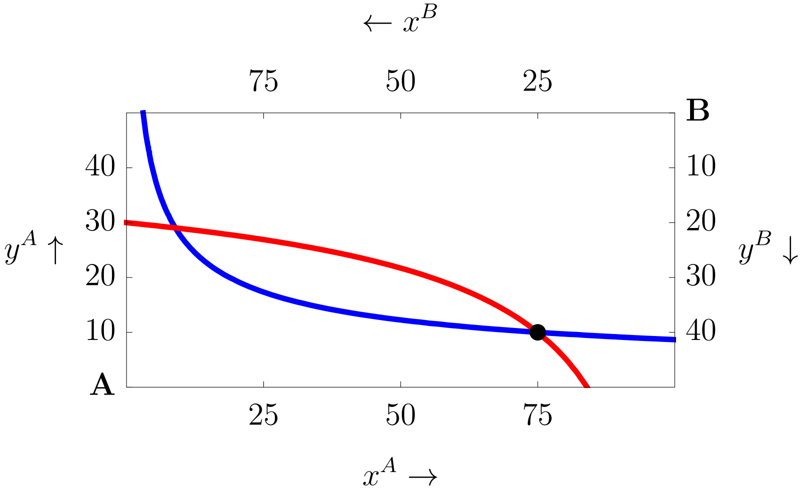
Let's demonstrate this with an example. Suppose we have two agents and two goods in an exchange economy.

- $e^A = (75, 10)$ and $e^B = (25, 40)$
- $u^A(x, y) = u^B(x, y) = \ln(x) + 2\ln(y)$

OK – here we go . . .

Edgeworth boxes (2)

Blue line = Anne's IC. Red line = Bob's IC. Point = endowment.



Edgeworth boxes (3)

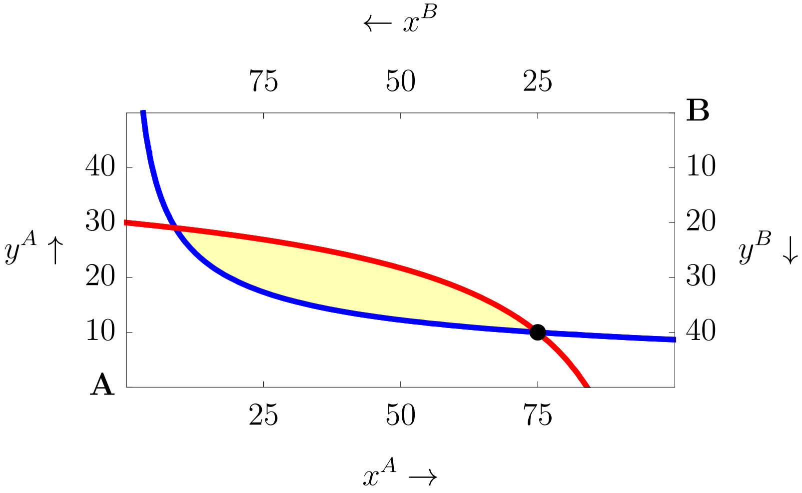
Is the endowment Pareto efficient?

- No. Remember Pareto efficiency implies that the two agents have the same MRS at a Pareto efficient allocation.
- We could do the math, but pretty obvious slopes of red and blue lines (the ICs) are different through the endowment.

Allocations in the shaded region are Pareto improvements...

Edgeworth boxes (4)

Blue line = Anne's IC. Red line = Bob's IC. Point = endowment.



Edgeworth boxes (5)

So what points are Pareto efficient? We can solve for them:

$$MRS^A_{XY}(x^A, y^A) = MRS^B_{XY}(x^B, y^B)$$
$$\frac{1/x^A}{2/y^A} = \frac{1/x^B}{2/y^B}$$

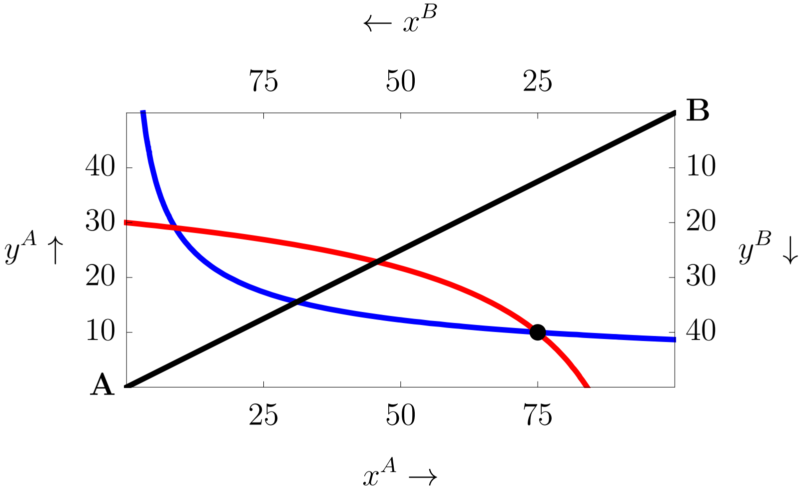
But note that $x^B = 100 - x^A$ and $y^B = 50 - y^A$ (by feasibility).

$$\frac{1/x^A}{2/y^A} = \frac{1/(100 - x^A)}{2/(50 - y^A)}$$
$$y^A = \frac{x^A}{2}$$

We call the set of Pareto efficient points the **contract curve**.

Edgeworth boxes (6)

Black line is the contract curve.



Edgeworth boxes (7)

OK – now let's **solve for the equilibrium** of this exchange economy and the equilibrium prices.

This is hard! Let's do it in steps.

Step 1: Normalize $p_x = 1$.

- What matters is relative prices, so we can just set one of them and work out the other.

Edgeworth boxes (8)

Step 2: Find incomes w^A and w^B.

$$w^A = p_x \cdot e_x^A + p_y \cdot e_y^A = 75 + 10p_y$$
$$w^B = p_x \cdot e_x^B + p_y \cdot e_y^B = 25 + 40p_y$$

Edgeworth boxes (9)

A little trick: If utility is $u^i(x^i, y^i) = a\ln(x^i) + b\ln(y^i)$ and agent i has income w^i, her demand function is:

$$x^i = \frac{a}{a+b}\frac{w^i}{p_x} \text{ and } y^i = \frac{b}{a+b}\frac{w^i}{p_y}$$

Step 3: Find demand functions for y.

$$y^A = \frac{b}{a+b}\frac{w^i}{p_y} = \frac{2}{3}\left(\frac{75+10p_y}{p_y}\right)$$

$$y^B = \frac{b}{a+b}\frac{w^i}{p_y} = \frac{2}{3}\left(\frac{25+40p_y}{p_y}\right)$$

Edgeworth boxes (10)

Step 4: Use market-clearing to find p_y:

$$\begin{aligned}
y^A + y^B &= e_y^A + e_y^B \\
\frac{2}{3}\left(\frac{75 + 10p_y}{p_y}\right) + \frac{2}{3}\left(\frac{25 + 40p_y}{p_y}\right) &= 50 \\
75 + 10p_y + 25 + 40p_y &= 75p_y \\
25p_y &= 100 \\
p_y &= 4
\end{aligned}$$

Edgeworth boxes (11)

Now we know the prices and the utility functions, so we have everything we need to find demand. We can find it just for Anne because Bob gets whatever is left:

Step 5: Plug $p_y = 4$ into the demand function to get y^A:

$$y^A = \frac{b}{a+b}\frac{w^i}{p_y} = \frac{2}{3}\left(\frac{75+10p_y}{p_y}\right) = \frac{2}{3}\left(\frac{75+40}{4}\right) = \frac{115}{6}$$

Edgeworth boxes (12)

Step 6: Plug y^A and p_y into BC to get x^A.

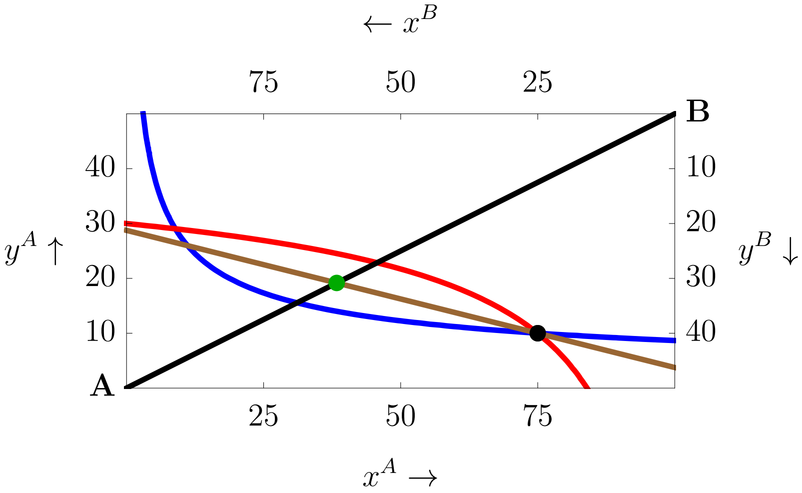
$$p_x x^A + p_y y^A = w^A$$
$$x^A + 4 \cdot \frac{115}{6} = 75 + 10 \cdot 4$$
$$x^A = \frac{115}{3}$$

Finally, we can plot the equilibrium allocation:

$$(x^A, y^A) = \left(\frac{115}{3}, \frac{115}{6}\right)$$

Edgeworth boxes (13)

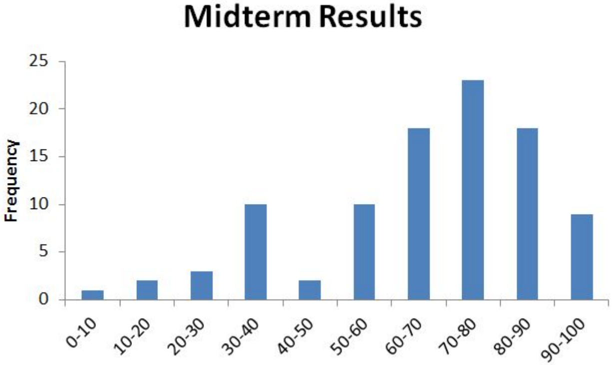
Green point is equilibrium. Brown line is budget constraint.



Where are we now?

1. Logarithmic utility = Cobb-Douglas utility
2. Exchange economies and edgeworth boxes
3. Reviewing the midterm

Exam results (1)



Percentile	10	20	30	40	50	60	70	80	90
Score	35.5	57	61	68	71	75	77.5	84	90

Mean = 67.

Exam results (2)

Note that letter grades have no meaning until the end of the course, when the problem sets, midterm, and final are combined into an aggregate raw score, and then letters are assigned to that aggregate through a curve.

Having said that, I know some of you will ask for rough letter grades for this exam. Scores correspond roughly as follow:

- $[84, 100] \approx A$
- $[78, 84) \approx A-$
- $[73, 78) \approx B+$
- $[65, 73) \approx B$
- $[60, 65) \approx B-$
- $[50, 60) \approx C+$
- $[35, 50) \approx C$
- $[30, 35) \approx C-$
- $[0, 30) \approx D+$ to F

Exam results (3)

Average score on each question:

Question	1(15)	2(12)	3(20)	4(16)	5(21)	6(8)	7(8)
Av. Score	11	7	12.6	11.4	14.3	6.2	4.8

Numbers in parentheses represent what each problem was worth.